COMBINATORY ANALYSIS

BY

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INTRODUCTION

VOLUME I of this work will not be found to overlap Netto's *Combinatorik* to any considerable extent.

The latter excellent treatise treats for the most part of combinations of objects which are all different, and deals with many special investigations of interest concerning them. It is particularly good in the historical treatment, and the reader who desires a comprehensive account of combinatory analysis should read the *Combinatorik*, as well as the present book. Investigations by, amongst other writers, Weyrauch, Longchamp, Tait, Cantor, Terquem, Weiss, Metzler, Sprague, Scheck, Schröder, Steiner, Reiss, Moore, De Vries, Heffter, Kirkman, Lucas, are given by Netto, which are valuable, but are not within the scope of the task undertaken here. This has in view the presentation of processes of great generality, and of new ideas, which have not up to the present time found a place in any book in any language.

This second volume commences with a detailed account of what is known concerning the algebra of the Theory of Partitions of Numbers, the point of view being that of Euler, and the first Section overlaps the *Combinatorik* to some extent. It does not include the investigations of Sylvester and Cayley in regard to expressions for the general coefficients in the enumerating generating functions. These are omitted as being arithmetical rather than algebraical. The reader is recommended to consult the original memoirs, and also a short account in the work of Netto.

New ideas will be found in Section VII in regard to:

(i) the transformation of certain series by a graphical method;

(ii) the discussion of Ramanujan's important identities;

(iii) the transformation of certain generating functions by means of symmetric functions;

(iv) the connexion of the theory of partitions with other combinatory theories.

The most important part of the volume, Sections VIII *et seq.*, arises from basing the theory of partitions upon the theory of Diophantine inequalities. This method is more fundamental than that of Euler, and leads directly to a high degree of generalization of the theory of partitions, and to several

investigations which are grouped together under the title of "Partition Analysis." A partition of a number consists essentially of an assemblage of integers whose sum is equal to the number partitioned. There is *à priori* no definite order amongst the integers (parts of the partition), and it therefore becomes possible to import a new idea into the definition of a partition by regarding a descending order of magnitude amongst the parts as essential. A partition thus becomes a series of integers in descending order of magnitude, and the problem of enumeration is transformed into the enumeration of the solutions of a set of Diophantine inequalities. In fact, a part of a partition is considered to have the attribute of position as well as of magnitude, the position being determined by relative magnitude.

The ordinary partitions of Section VII are regarded, from this point of view, as essentially belonging to space of one dimension with a graphical representation in two dimensions. One generalization, taken up in Sections IX and X, considers partitions which appertain essentially to space of two dimensions, with a graphical representation in three dimensions.

The complete solution of this problem is given. After fruitless researches extending over many years it finally yielded before the invention of the lattice function and its derivatives. This function, which is otherwise of much intrinsic interest, is formed from the lattice permutations of a given assemblage of letters through the medium of the index of a permutation. The assemblage of letters spoken of arises from the Ferrers graph, which involves the points or nodes at which the parts of a partition, subject to Diophantine inequalities, are placed. The lattice permutations, and the theory of the indices, were studied in Section III of Vol. I.

The subject of partitions in space of three dimensions is also broached, and the complete solution in regard to the summits of a cube is reached. There is here no convenient graphical representation, and the general question appears to bristle with difficulties.

In Section XI an account is given of the Theory of the Symmetric Functions of several systems of quantities. A knowledge of this subject must be of service in further researches in Combinatory Analysis. It is here shewn to yield a second brief and elegant solution of the question of the Latin Square and its generalizations.

In conclusion, I would say that I am aware that the reader will probably find imperfections in these volumes, but I shall be satisfied if they are found to contain ideas which are new and fresh, and such as are likely to prove starting-points for further investigations in an exceedingly interesting field of pure mathematics.

I wish to express my thanks to the Cambridge University Press for the care they have taken in producing the work, and to Mr H. B. C. Darling, of the Standards Department of the Board of Trade, for his kindness in reading the proofs of both volumes.

NOTE ON WARING'S FORMULA FOR THE SUM OF THE POWERS OF THE ROOTS OF AN EQUATION

In the work *Grundzüge der Antiken und Modernen Algebra* von Ludwig Matthiessen, Leipzig, 1878, there appears, p. 62, the following statement:

"Der niederländische Mathematiker Albert Girard hat in einer 1629 verfassten Schrift für die Potenzsumme die Formel

$$S_m = m\Sigma \frac{(-)^{s_1+s_2+\ldots+s_m}(s_1+s_2+\ldots+s_n-1)!\, a^{s_1}b^{s_2}\ldots t^{s_n}}{s_1!\, s_2! \ldots s_n!}$$

angegeben. Sie wird zumeist Waring zugeschrieben, der sie erst 1782 und zwar ohne Beweis mittheilt";

with a foot-note:

"Girard, *Invention nouvelle en l'Algèbre*, Amsterdam, 1629."

This statement is absolutely incorrect, as I will now shew. I was unfortunately misled by it to ascribe the formula in Vol. I of this work to Girard.

Girard, in the pamphlet quoted by Matthiessen, gives the formulæ

$$A$$

$$Aq - B2$$

$$Acub - AB3 + C3$$

$$Aqq - AqB4 + AC4 + Bq2 - D4$$

for the sums of the first, second, third and fourth powers of the roots. He gives no formula for the general coefficient, and there is no sign whatever that he knew of the existence of such a formula.

So much for Matthiessen's first statement.

In regard to Waring, the theorem in question forms the subject of the first problem in Chapter I of the *Meditationes Algebraicæ*, which was published at Cambridge in 1770, and therein is given a complete proof by mathematical induction.

Compare Matthiessen's second statement to the effect that Waring first gave the formula in 1782, and then without proof. It appears, therefore, that the theorem is due to Waring, and I much regret having failed to verify Matthiessen's references before adopting his view.

I am indebted to Dr R. F. Muirhead for drawing my attention to this subject, and for giving me the reference to his paper, "A proof of Waring's Expression for $\Sigma\alpha^r$ in terms of the Coefficients of an Equation," *Proc. Edin. Math. Soc.* Vol. XXIII., wherein the matter is discussed.

NOTE ON THE GREATER INDEX OF A PERMUTATION AND THE NUMBER OF INVERSIONS OF A PERMUTATION

It is shewn in Vol. II, Sect. IX, p. 206 that, in regard to any assemblage of letters, if p be the greater index of a permutation and we take Σx^p, the summation being in regard to every permutation of the assemblage,

$$\Sigma x^p = \frac{(1)(2)\dots(i+j+k+\dots)}{(1)(2)\dots(i).(1)(2)\dots(j).(1)(2)\dots(k)\dots},$$

where the assemblage of letters is taken to be

$$\alpha^i\beta^j\gamma^k\dots.$$

Netto and other investigators have on the other hand considered the number of inversions of a permutation, say v.

I have recently shewn in *Proc. L. M. S.* 1916 that while p and v are different numbers for the same permutation, the assemblage of numbers p is identical with the assemblage of numbers v. Hence

$$\Sigma x^v = \Sigma x^p,$$

and the properties of the numbers v are comprised in the algebraic expression of Σx^p given above.

Compare Netto's *Combinatorik*, p. 92, and this work, Vol. I, p. 135.

To the original papers reference is made in regard to:

Section VII. Sylvester, *Collected Mathematical Papers*, Vol. III.:
pp. 249–51, "Note on continuants";
pp. 653–5, "On Crocchi's theorem";
pp. 658–60, "On the fundamental theorem...";
pp. 661–3, "Note on the paper by Mr Durfee...";
pp. 664–6, "Note on Dr Franklin's proof...";
pp. 667–71, "On the use of cross gratings...";
pp. 677–9, "Proof of a well-known development...";
pp. 680–2, "On a new theorem in partitions";
pp. 683–4, "Note on the graphical method in partitions";
pp. 685–6, "An instantaneous graphical proof...";
and in Vol. IV.:
pp. 1–83, "A constructive theory of partitions... ."

Cayley, *Collected Mathematical Papers*, Vol. II. pp. 235–49, 506–12, "Researches in the partition of numbers."

The Author, *Phil. Trans. R. S.* 1896, A, "Memoir on the theory of the partition of numbers—Part I."

S. Ramanujan.

Section VIII. *Phil. Trans. R. S.* 1899, A, "Memoir on the theory of the partition of numbers—Part II." and 1905, A,—Part III.

Camb. Phil. Trans. Vol. XVIII. "Application of the partition analysis to the study of the properties of any system of consecutive integers."

Camb. Phil. Trans. Vol. XIX. Part i. "The Diophantine inequality $\lambda x \geqslant \mu y$."

Section IX. *Phil. Trans. R. S.* 1911, A, "Memoir on the theory of the partition of numbers—Part V. Partitions in two-dimensional space."

Section X. *Phil. Trans. R. S.* 1911, A, "Memoir on the theory of the partition of numbers—Part VI."

Camb. Phil. Trans. Vol. XVII. Part ii. "Partitions of numbers whose graphs possess symmetry."

Section XI. *Phil. Trans. R. S.* 1890, A, "Memoir on symmetric functions of roots of systems of equations."

P. A. M.

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THE THEORY OF EULER

238. We have now to consider in interesting detail the division of an integer into other integers of which it is the sum. The integers last mentioned are the parts of the partition of the first mentioned integer. These parts may be drawn from the whole series of natural numbers or they may be drawn from a system of integers which is duly specified; thus the system may be that which involves the uneven integers or the even integers exclusively, or the actual numbers which may appear as parts may be particularly specified as being any integers agreed upon. Further the number of times which each integer or the whole of the integers may appear as parts may be fixed and also the whole number of the parts of the partition. The part magnitude may also be fixed and a host of other conditions may be imposed upon the partitions.

Euler's theory proceeds from the fact that algebraical multiplication of integral powers of the same numerical magnitude is equivalent to the arithmetical addition of the powers; thus

$$x^{p_1} . x^{p_2} . x^{p_3} \ldots = x^{p_1+p_2+p_3+\ldots}.$$

It thence follows at once that the whole number of partitions of an integer n is equal to the coefficient of x^n in the expansion of the function

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots \text{ad inf.}}.$$

Ex. gr. expanding each factor into an infinite series and then multiplying them together we find that the term $7x^5$ arises made up in the multiplication in the manner

$$x^{1+1+1+1+1} + x^{1+1+1+2} + x^{1+2+2} + x^{1+1+3} + x^{2+3} + x^{1+4} + x^5$$

due to the expansion

$$(1-x^r)^{-1} = 1 + x^r + x^{2r} + x^{3r} + \ldots,$$

where r has all values not exceeding in this instance 5.

The above written algebraic fraction we call the enumerating generating function (G.F. for brevity) of the partitions of n. It is also clear that if the part magnitude be restricted to not exceed j, the enumerating generating function will be

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^j)}.$$

If the condition of partition be that there are to be exactly i parts the number of partitions of n will be equal to the coefficient of $a^i x^n$ in the expanded fraction

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)\dots \text{ad inf.}}.$$

This fraction may be expanded in ascending powers of a in many ways; one of the simplest is to write it as a function $F(a)$ of a and to put

$$F(a)=1+a\phi_1(x)+a^2\phi_2(x)+a^3\phi_3(x)+\dots,$$

from which we derive

$$F(ax)=1+ax\phi_1(x)+a^2x^2\phi_2(x)+a^3x^3\phi_3(x)+\dots.$$

Now, from the G.F.,

$$F(ax)=(1-ax)F(a),$$

so that

$$1+ax\phi_1(x)+a^2x^2\phi_2(x)+\dots=(1-ax)\{1+a\phi_1(x)+a^2\phi_2(x)+\dots\}.$$

Herein equating coefficients of like powers of a we find

$$\begin{aligned} x\phi_1(x) &= \phi_1(x)-x, \\ x^2\phi_2(x) &= \phi_2(x)-x\phi_1(x), \\ x^3\phi_3(x) &= \phi_3(x)-x\phi_2(x), \\ &\text{etc.} \end{aligned}$$

from which

$$\phi_1(x)=\frac{x}{1-x},$$

$$\phi_2(x)=\frac{x^2}{(1-x)(1-x^2)},$$

and in general

$$\phi_i(x)=\frac{x^i}{(1-x)(1-x^2)\dots(1-x^i)},$$

leading to the algebraical identity

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)\dots \text{ad inf.}}$$

$$=1+\frac{ax}{1-x}+\frac{a^2x^2}{(1-x)(1-x^2)}+\dots+\frac{a^ix^i}{(1-x)(1-x^2)\dots(1-x^i)}+\dots,$$

and to the conclusion that the enumerating G.F. for partitions into exactly i parts is

$$\frac{x^i}{(1-x)(1-x^2)\dots(1-x^i)},$$

the number required being given by the coefficient therein of x^n.

239. Moreover this coefficient is equal to the coefficient of x^{n-i} in the expanded fraction

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^i)},$$

a fact which demonstrates that n has as many partitions which involve exactly i parts as $n-i$ has partitions in which the part magnitude is limited so as not to exceed i. This theorem is not obvious immediately *à priori* but becomes so on further consideration. Ferrers (and after him notably Sylvester) denoted a partition of a number graphically. The parts being arranged in descending order of magnitude from left to right, corresponding rows of dots or nodes are placed in succession, each row starting from the same line parallel to the long edge of the page; thus the graph of a partition 7521 will be

$$\begin{array}{lllllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & & & & & \\
\bullet & & & & & &
\end{array}$$

wherein it will be noticed that, in general, the number of rows is equal to the number of parts in the partition while the number of columns is equal to the highest part. If we now rotate this graph so that the rows become columns we find the graph

$$\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & & \\
\bullet & \bullet & & \\
\bullet & \bullet & & \\
\bullet & & & \\
\bullet & & &
\end{array}$$

which now denotes the partition 4322211 of the same number 15. The two partitions are said to be conjugate; and so also the two graphs.

It is clear that if a graph denote a partition having a highest part j and exactly i parts, the conjugate graph will denote one which has a highest part i and exactly j parts. Hence these two sets of partitions are equinumerous. Moreover if we have a set of partitions in which the highest part is limited not to exceed j, the conjugate set of partitions will be such that the number of parts is limited not to exceed j. We thus see clearly

that the partitions having a highest part not exceeding j have the same enumerating G.F. as the partitions in which the number of parts does not exceed j. The theorem which gave rise to this discussion may therefore be stated in the form: "A number n has as many partitions involving exactly i parts as the number $n-i$ has partitions which do not involve more than i parts." In this form its truth is seen at once, for if we diminish each of the parts of the former partitions by unity we obtain each of the latter partitions.

Ex. gr. $n=11$, $i=3$.

The sets of partitions are

911	821	731	722	641	632	551	542	533	443
8	71	62	611	53	521	44	431	422	332

Conversely reasoning in this manner leads at once to the expansion of the reciprocal of $(1-ax)(1-ax^2)(1-ax^3)\ldots$ ad inf. in ascending powers of a.

240. We can obtain a different expansion of the same fraction from the following consideration. The G.F. of the partitions which have exactly i parts and a highest part precisely equal to j is clearly

$$\frac{ax^j}{(1-ax)(1-ax^2)\ldots(1-ax^j)},$$

the number of partitions being given by the coefficient of $a^i x^n$.

Hence giving j all values

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)\ldots\text{ ad inf.}}$$

$$=1+\frac{ax}{1-ax}+\frac{ax^2}{(1-ax)(1-ax^2)}+\frac{ax^3}{(1-ax)(1-ax^2)(1-ax^3)}+\ldots\text{ ad inf.}$$

The partitions which involve not more than i parts are given by the coefficient of $a^i x^n$ in the G.F.

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\ldots\text{ ad inf.}}$$

$$=1+\frac{a}{1-x}+\frac{a^2}{(1-x)(1-x^2)}+\frac{a^3}{(1-x)(1-x^2)(1-x^3)}+\ldots.$$

This series is obtained as before from the consideration that writing ax for a is equivalent to multiplication by $1-a$.

The number in question is the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^i)},$$

verifying a previous conclusion.

241. We now come to the important G.F. which enumerates the partitions in which the number of parts and the highest part are limited by the numbers i, j respectively. The number is clearly given by the coefficient of $a^i x^n$ in the fraction

$$\frac{1}{(1-a)(1-ax)(1-ax^2)\dots(1-ax^j)}.$$

Writing ax for a is equivalent to multiplication by

$$\frac{1-a}{1-ax^{j+1}};$$

hence as before we are at once led to the expanded form

$$1 + a\frac{1-x^{j+1}}{1-x} + a^2\frac{(1-x^{j+1})(1-x^{j+2})}{(1-x)(1-x^2)} + \dots + a^i\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{i+j})}{(1-x)(1-x^2)\dots(1-x^i)} + \dots,$$

establishing that the enumerating G.F. is

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{i+j})}{(1-x)(1-x^2)\dots(1-x^i)},$$

which may be also written

$$\frac{(1-x)(1-x^2)\dots(1-x^{i+j})}{(1-x)(1-x^2)\dots(1-x^i).(1-x)(1-x^2)\dots(1-x^j)}.$$

This is symmetrical in i and j, as should be the case from the consideration of conjugate graphs*.

Cayley, in his researches on partitions, adopted a very convenient notation. He wrote

$$1 - x^m = (\mathbf{m}),$$

thus the above generating function would be written

$$\frac{(\mathbf{j+1})(\mathbf{j+2})\dots(\mathbf{i+j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i})}.$$

242. It is important at this stage to examine the G.F.

$$\frac{x^i}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})}$$

in some detail so as to become familiar with a method of interpretation in the theory of partitions.

We take the particular case

$$\frac{x^9}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}.$$

The highest part of a partition exceeds the next highest part by one of the numbers 0, 1, 2, ... n. This will be called the "first excess"; similarly the

* An arithmetical proof of this theorem has been given by Franklin, *Amer. Journ. of Math.* Vol. IV. p. 18.

second highest part (which may be equal to the highest part) exceeds the next occurring part by some number which we shall call the "second excess," and so on.

Take any partition of 9 into three or fewer parts and express it graphically; thus suppose we take the partition 333

$$\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \circ & \circ & & & \\
\bullet & \bullet & \bullet & \circ & & & &
\end{array}$$

and continue it by any graph of three rows represented by hollow dots or nodes. We may suppose the G.F. to represent such graphs where the filled nodes are fixed and the hollow nodes can be written down at pleasure so as to constitute the regular graph of a partition. The partitions represented by the combined graph are specified by the statement that "there are exactly three parts and no part is less than three." If instead of 333 we take the partition 531 we find

$$\begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & & & \circ & \circ & & & \\
\bullet & & & & & \circ & & & &
\end{array}$$

and the hollow nodes must be regarded as moved to the left as far as possible so as to form the combined regular graph

$$\begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \circ & \circ & & & & & \\
\bullet & \circ & & & & & & & &
\end{array}$$

We look upon the filled-in nodes as fixed and the hollow ones to be assigned at pleasure. We have here exactly three parts with the conditions

(i) the first excess must be at least 2,

(ii) the second excess must be at least 2.

These partitions are therefore also enumerated by the G.F.

The conditions in regard to the first two excesses shew that the partitions under view are those in which there are no repetitions or sequences of parts. In fact consecutive parts such as p, p; $p+1, p$; are clearly ruled out.

Bringing under view the successive partitions

$$1,\ 31,\ 531,\ 7531,\ 97531, \ldots,$$

we see that the partitions of numbers which involve neither repetitions nor sequences are enumerated by the series

$$1 + \frac{x}{(1)} + \frac{x^4}{(1)(2)} + \frac{x^9}{(1)(2)(3)} + \frac{x^{16}}{(1)(2)(3)(4)} + \cdots,$$

the powers of x involved in the numerators being the successive square numbers.

Again if we take the partition 72 of 9 the partitions enumerated will be those into not less than two nor more than three parts subject to the condition that the first and second excesses must be at least 5 and 2 respectively.

Generally, in respect of the fraction

$$\frac{x^i}{(1)(2)(3)\dots(j)},$$

considering the partition $p_1 p_2 p_3 \dots p_s$ of i the partitions enumerated will be those which involve not less than s nor more than j parts subject to the condition that the μth excess must be at least $p_\mu - p_{\mu-1}$, μ having all values from 2 to s.

243. Another interesting series is obtained by consideration of the successive partitions

1, 21, 321, 4321, etc. ....

This is

$$1 + \frac{x}{(1)} + \frac{x^3}{(1)(2)} + \frac{x^6}{(1)(2)(3)} + \frac{x^{10}}{(1)(2)(3)(4)} + \dots,$$

the exponents that occur in the numerators being the triangular numbers. The series clearly enumerates the partitions in which no part is repeated; in other words partitions into unequal numbers. Moreover such partitions are obviously enumerated by the infinite product

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\dots \text{ ad inf.}$$

Hence the identity

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\dots$$

$$= 1 + \frac{x}{(1)} + \frac{x^3}{(1)(2)} + \frac{x^6}{(1)(2)(3)} + \frac{x^{10}}{(1)(2)(3)(4)} + \dots.$$

244. An important identity is

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\dots.$$

This is obvious as soon as we multiply up by $1-x$.

On interpretation it establishes that every integer can be partitioned in one and only in one way into parts which are different powers of 2.

There is thus one partition uniquely connected with every integer. The parts are different powers of 2. The reader will have no difficulty in establishing that a particular power of 2, say 2^α, will or will not be a part in this special partition of an integer n according as the greatest integer in $n/2^\alpha$ is uneven or even.

For example, since the greatest integer in 73/8 is 9, an uneven number, 2^3 will be one part in the special partition of 73. The partition in question is in fact $64 + 8 + 1$.

Stated in another way we may say that

$$\Sigma 2^a = n,$$

where the summation is for every integer a such that

$$E\, n/2^a$$

is uneven.

Writing x^m for x we find

$$\frac{1}{1-x^m} = (1+x^m)(1+x^{2m})(1+x^{4m})(1+x^{8m})\ldots,$$

a formula of use in the transformation of generating functions. Thus

$$\begin{aligned}
&\frac{1}{(1)\,(3)\,(5)\,(7)\ldots}\\
&=(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\\
&\times(1+x^3)(1+x^6)(1+x^{12})(1+x^{24})\ldots\\
&\times(1+x^5)(1+x^{10})(1+x^{20})(1+x^{40})\ldots\\
&\times(1+x^7)(1+x^{14})(1+x^{28})(1+x^{56})\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&=(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots
\end{aligned}$$

because every integer is uniquely expressible as the product of an uneven number and a power of 2.

If we apply the transformation to the G.F.

$$\frac{1}{(1)\,(2)\,(3)\,(4)\ldots},$$

it becomes

$$\begin{aligned}
&(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ldots\\
&\times(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ldots\\
&\times(1+x^3)(1+x^6)(1+x^{12})\ldots\\
&\times(1+x^4)(1+x^8)(1+x^{16})\ldots\\
&\times(1+x^5)(1+x^{10})\ldots\\
&\times\ldots\\
&\quad\ldots
\end{aligned}$$

and in this infinite product $1 + x^m$ occurs with an exponent $1 + p$ where 2^p is the highest power of 2 which is a factor of m.

We may therefore write the G.F.

$$(1+x)(1+x^2)^2(1+x^3)(1+x^4)^3(1+x^5)(1+x^6)^2(1+x^7)(1+x^8)^4\ldots,$$

and interpret it to mean that the whole number of partitions of n is equal to the number of partitions into the parts

$$1,\ 2_1,\ 2_2,\ 3,\ 4_1,\ 4_2,\ 4_3,\ 5,\ 6_1,\ 6_2,\ 7,\ 8_1,\ 8_2,\ 8_3,\ 8_4,\ \dots$$

where 4_1, 4_2, 4_3, for example, denote the number 4 in three different colours and the partitions are such that no number occurs twice as a part with the same colour. The seven partitions of the number 5 are

$$5,\ 4_1 1,\ 4_2 1,\ 4_3 1,\ 3 2_1,\ 3 2_2,\ 2_1 2_2 1.$$

245. The identity

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8) + \dots \text{ ad inf.}$$

may be put into the form

$$\frac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\dots \text{ ad inf.}} = 1 - x,$$

from which expansion of the left-hand side shews that every number greater than unity when partitioned into repeatable parts which are powers of 2 has as many partitions involving an even number of parts as partitions involving an uneven number. As an example, the number 12 has the partitions

Even number of parts	Uneven number of parts
$84,\ 821^2,\ 4^2 2^2,\ 4^2 1^4,\ 42^3 1^2,$	$82^2,\ 81^4,\ 4^3,\ 4^2 21^2,\ 42^4,$
$421^6,\ 2^6,\ 2^4 1^4,\ 2^2 1^8,\ 1^{12},$	$42^2 1^4,\ 41^8,\ 2^5 1^2,\ 2^3 1^6,\ 21^{10},$

the number being ten in each case.

246. The expansion of the finite product

$$(1-ax)(1-ax^2)\dots(1-ax^i)$$

can be derived from the expansion of its reciprocal by the following considerations. The coefficient of $a^j x^n$ in the former is equal to the number of partitions of n into exactly j unequal parts, none greater than i; the same coefficient in the latter is equal to the number of partitions of n into exactly j parts, equal or unequal, the part magnitude as before being limited so as not to exceed i.

Now if one of the latter partitions written in ascending order of part magnitude be

$$p_1,\ p_2,\ p_3,\ \dots\ p_j,$$

where
$$p_1 + p_2 + p_3 + \dots + p_j = n,$$

it is clear that

$$p_1 + 0,\ p_2 + 1,\ p_3 + 2,\ \dots\ p_j + j - 1$$

is a partition of $n + \binom{j}{2}$ into exactly j unequal parts, the limit to the part magnitude being $i + j - 1$.

Conversely from any partition of $n+\binom{j}{2}$ into exactly j unequal parts, no part being greater than $i+j-1$, we can by subtracting the numbers $0, 1, 2, \ldots j-1$ from the successive parts reach a partition of n into exactly j equal or unequal parts, no part greater than i. There is thus a one-to-one correspondence between the system of partitions of n into exactly j equal or unequal parts, no part greater than i, and the system of partitions of $n+\binom{j}{2}$ into exactly j unequal parts, no part greater than $i+j-1$. Therefore we pass to the coefficient of a^j in the expansion of

$$(1-ax)(1-ax^2)\ldots(1-ax^i)$$

from the coefficient of a^j in the expansion of

$$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^i)}$$

by changing i into $i-j+1$ and multiplying by $x^{\binom{j}{2}}$.

Since therefore $$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^i)}$$

$$=1+\frac{a(1-x^i)x}{1-x}+\frac{a^2(1-x^i)(1-x^{i+1})x^2}{(1-x)(1-x^2)}+\ldots$$

$$+\frac{a^j(1-x^i)(1-x^{i+1})\ldots(1-x^{i+j-1})x^j}{(1-x)(1-x^2)\ldots(1-x^j)}+\ldots,$$

we find that $$(1-ax)(1-ax^2)\ldots(1-ax^i)$$

$$=1-\frac{a(1-x^i)x}{1-x}+\frac{a^2(1-x^{i-1})(1-x^i)x^3}{(1-x)(1-x^2)}-\ldots$$

$$+(-)^j\frac{a^j(1-x^{i-j+1})(1-x^{i-j+2})\ldots(1-x^i)x^{\binom{j+1}{2}}}{(1-x)(1-x^2)\ldots(1-x^j)}+\ldots.$$

The generating function for partitions into exactly j unequal parts none greater than i is therefore

$$\frac{(i-j+1)(i-j+2)\ldots(i)}{(1)(2)\ldots(j)}x^{\binom{j+1}{2}},$$

or as it may be also written

$$\frac{(j+1)(j+2)\ldots(i)}{(1)(2)\ldots(i-j)}x^{\binom{j+1}{2}}.$$

247. Let us next consider partitions into uneven parts.

The G.F. is clearly $$\frac{1}{(1)(3)(5)(7)(9)\ldots}.$$

Multiplying numerator and denominator by

$$(2)\,(4)\,(6)\,(8)\,(10)\ldots,$$

we obtain
$$\frac{(2)\,(4)\,(6)\,(8)\,(10)\ldots}{(1)\,(2)\,(3)\,(4)\,(5)\ldots},$$

which is
$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots,$$

which is none other than the G.F. for partitions into unequal parts. We thus find the theorem: "Any number can be partitioned into uneven parts in the same number of ways as it can be partitioned into unequal parts." Cf. Art. 243.

Again, putting

$$F(a)=\frac{1}{(1-ax)(1-ax^3)(1-ax^5)(1-ax^7)\ldots\text{ ad inf.}},$$

we find that
$$F(ax^2)=(1-ax)\,F(a),$$

and we are led to the expansion

$$F(a)=1+\frac{ax}{(2)}+\frac{a^2x^2}{(2)\,(4)}+\frac{a^3x^3}{(2)\,(4)\,(6)}+\ldots.$$

This shews that the G.F. which enumerates the partitions into exactly i uneven integers is

$$\frac{x^i}{(2)\,(4)\,(6)\ldots(2i)},$$

a result which might have been predicted, because considering i nodes placed in a column

$$\begin{array}{c}\bullet\\ \bullet\\ \bullet\\ \bullet\\ \vdots\end{array}$$

if we place to the right of it the graph of any partition which involves i even parts and no uneven parts we find that the combined graph exhibits a partition composed of exactly i uneven parts.

248. In a precisely similar manner we are led to the expansion

$$\frac{1}{(1-ax)(1-ax^3)\ldots(1-ax^{2j-1})}$$
$$=1+ax\frac{(2j)}{(2)}+a^2x^2\frac{(2j)\,(2j+2)}{(2)\,(4)}+\ldots+a^ix^i\frac{(2j)\,(2j+2)\ldots(2i+2j-2)}{(2)\,(4)\ldots(2i)}+\ldots,$$

establishing that the G.F. for partitions into uneven numbers such that the highest part does not exceed $2j-1$ while there are exactly i parts is

$$x^i\frac{(2j)\,(2j+2)\ldots(2i+2j-2)}{(2)\,(4)\ldots(2i)}.$$

The fact that the partitions into uneven parts are equi-numerous with those into unequal parts has been shewn by Glaisher in an ingenious manner. He establishes a one-to-one correspondence between the two sets of partitions. Writing a partition into uneven numbers in the form

$$1^{\alpha_1} 3^{\alpha_3} 5^{\alpha_5} 7^{\alpha_7} \ldots,$$

$\alpha_1, \alpha_3, \alpha_5, \alpha_7, \ldots$ denoting repetitions of parts, he expresses these numbers α in the binary scale; thus

$$\alpha_{2s+1} = 2^{\beta_{1s}} + 2^{\beta_{2s}} + 2^{\beta_{3s}} + \ldots,$$

and thence proceeds to the partition

$$1 \times (2^{\beta_{10}} + 2^{\beta_{20}} + \ldots) + 3 \times (2^{\beta_{11}} + 2^{\beta_{21}} + \ldots) + 5 \times (2^{\beta_{12}} + 2^{\beta_{22}} + \ldots) + \ldots,$$

the parts being

$$2^{\beta_{10}},\ 2^{\beta_{20}}, \ldots\ 3 \,.\, 2^{\beta_{11}},\ 3 \,.\, 2^{\beta_{21}}, \ldots\ 5 \,.\, 2^{\beta_{12}},\ 5 \,.\, 2^{\beta_{22}}, \ldots,$$

and asserts that these parts are all different because any integer is only expressible in one way as the product of an uneven integer and a power of 2.

The correspondence in the case of the number 9 is

9	71^2	531	51^4	3^3	$3^2 1^3$	31^6	1^9	parts uneven,
9	72	531	54	63	621	432	81	parts unequal.

In this connexion it should be observed that the function which enumerates partitions into uneven parts, viz.:

$$\frac{1}{(1)(3)(5)(7)\ldots},$$

may be put into the form (by Art. 244)

$$\begin{aligned}
&(1 + x)(1 + x^2)(1 + x^{2^2})(1 + x^{2^3})\ldots \\
\times\, &(1 + x^3)(1 + x^{3 \cdot 2})(1 + x^{3 \cdot 2^2})(1 + x^{3 \cdot 2^3})\ldots \\
\times\, &(1 + x^5)(1 + x^{5 \cdot 2})(1 + x^{5 \cdot 2^2})(1 + x^{5 \cdot 2^3})\ldots \\
\times\, &(1 + x^7)(1 + x^{7 \cdot 2})(1 + x^{7 \cdot 2^2})(1 + x^{7 \cdot 2^3})\ldots \\
\times\, &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

wherein the sth row of factors is equal to $\dfrac{1}{(2s-1)}$.

Considering now the partition into uneven parts

$$1^{p_1} 3^{p_3} 5^{p_5} \ldots (2s-1)^{p_{2s-1}} \ldots,$$

it is clear that the factor $\qquad x^{p_{2s-1}(2s-1)}$

of the term $\qquad x^{p_1 . 1 + p_3 . 3 + p_5 . 5 + \ldots + p_{2s-1}(2s-1) + \ldots}$

is obtained from the sth row of factors in the form

$$x^{(2s-1)(2^{\alpha_1} + 2^{\alpha_2} + 2^{\alpha_3} + \ldots)},$$

and that we thus arrive at a partition of $p_{2s-1}(2s-1)$ into unequal parts Moreover varying s so as to obtain similarly a partition of

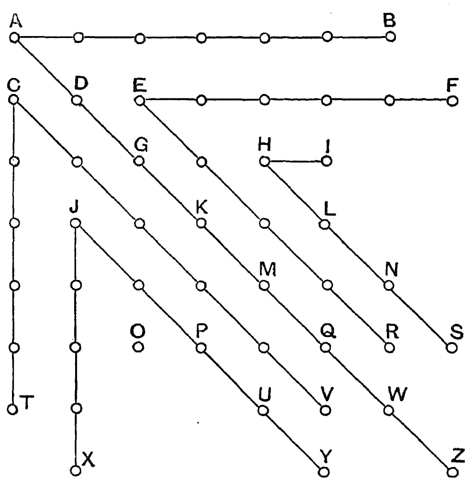
$$p_1 . 1 + p_3 . 3 + p_5 . 5 + \ldots + p_{2s-1}(2s-1) + \ldots$$

into parts we see that no two parts can be equal since we know otherwise that the enumerating function is identically the same as

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots.$$

Hence the form of enumerating function written above involves necessarily the one-to-one correspondence under discussion. It is also the analytical expression of the fact that a number is only expressible in one way as the product of an uneven number and a power of 2.

249. Another method of exhibiting a one-to-one correspondence is due to Sylvester. It is valuable as shewing the power of the graphical representation. The correspondence differs from that obtained by Glaisher. In the discussion of self-conjugate partitions (Art. 252) it will be shewn that the graph may be composed by means of angles of nodes, each angle containing an uneven number of nodes, and it will thus appear that the number of self-conjugate partitions of a number n is equal to the number of ways of composing n with unequal uneven numbers. In the present case we are given a partition of n into repeatable uneven numbers and we may form an angle graph, but it will not be regular.



Thus let the partition into uneven numbers be

13 13 7 5 5 5 1 1

of the partible number 50 and form the angle graph by making the angles

BAT, *FDX*, *IGO*, *LKP*, *NMU*, *SQY*, *W*, *Z*

to contain 13, 13, 7, 5, 5, 5, 1, 1 nodes respectively.

The graph is now to be read in the manner

$$ZAB,\quad VCT,\quad REF,\quad YJX,\quad SHI,\quad O,$$

yielding the partition into unequal numbers

$$14,\ 11,\ 10,\ 9,\ 5,\ 1$$

of the partible number 50.

Slight consideration shews that the process must produce a partition into unequal numbers.

The correspondence in the case of the number 9 yields

9	71^2	531	51^4	3^3	3^21^3	31^6	1^9
54	63	531	72	432	621	81	9

which should be contrasted with that reached by the method of Glaisher.

250. There is also a more refined correspondence arising from this point of view. A partition into unequal parts may involve s consecutive numbers. This is termed a sequence of order s. If a number is not accompanied by the next highest or the next lowest number this would be called a sequence of order *one*. This is obviously the same as no sequence at all, but in what follows it counts as a sequence. A partition into s unequal parts may therefore involve $1, 2, 3, \ldots$ or s sequences. For example the partition

$$14,\ 11,\ 10,\ 9,\ 5,\ 1$$

involves four sequences, viz.:

$$14;\ \ 11,\ 10,\ 9;\ \ 5;\ \ 1.$$

This partition was derived, by Sylvester's graphical method, from the partition

$$13,\ 13,\ 7,\ 5,\ 5,\ 5,\ 1,\ 1$$

which is composed of four distinct uneven numbers.

The correspondence to be noticed here is that between the partitions which are composed of s distinct uneven numbers and the partitions into unequal parts which involve s sequences.

Thus for the number 9 we have for

$s=1$,	$9,\ 3^3,\ 1^9$	uneven parts,
	$54,\ 432,\ 9$	unequal „
$s=2$,	$71^2,\ 51^4,\ 3^21^3,\ 31^6$	uneven „
	$63,\ 72,\ 621,\ 81$	unequal „
$s=3$,	531	uneven „
	531	unequal „

The proof of the theorem, too long to be given here, will be found in Sylvester's *Collected Mathematical Papers*, Vol. IV. p. 45.

CHAPTER II

GEOMETRICAL AND OTHER TRANSFORMATIONS

251. One of the most important methods of studying the graph of a partition is due to Durfee*. He fixed attention upon the square of nodes that appertains to every graph. Consider the graphs of the partitions 32, 662, 4431, viz.:

$$\begin{array}{lll} \circ & \circ & \bullet \\ \circ & \circ & \\ & & \\ & & \end{array} \qquad \begin{array}{llllll} \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & & & \\ & & & & & \end{array} \qquad \begin{array}{llll} \circ & \circ & \circ & \bullet \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \circ & \\ \bullet & & & \end{array}$$

where the square of nodes is specified in each case. The reader will observe that if the mth part of the partition is at least equal to m, and the $(m+1)$th part less than $m+1$, the graph involves a square of nodes having m nodes in the side. It follows at once that every graph of i rows and j columns can be dissected into three pieces:

(i) a square containing m^2 nodes,

(ii) a lateral graph of m rows (at most) and $j-m$ columns,

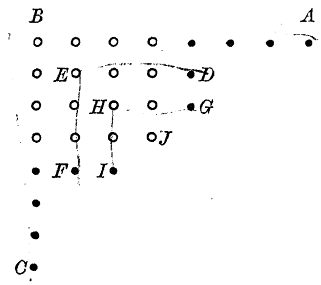
(iii) a subjacent graph of $i-m$ rows and m columns (at most).

The number m may be as large as the least of the numbers i, j, and may have any lesser value.

If the content of the graph be n, and there is a square of m^2 nodes, we may distribute $n-m^2$ nodes into two groups, one of which involves m or fewer rows and exactly $j-m$ columns, and the other exactly $i-m$ rows and m or fewer columns. The corresponding partitions will have in the one case m or fewer parts and a highest part equal to $j-m$, and in the other exactly $i-m$ parts and a highest part not greater than m.

* *Johns Hopkins University Circulars*, II. (1883); Sylvester's *Collected Mathematical Papers*, Vol. III. p. 661.

Many applications may be made of the dissection above described. Consider in the first place a self-conjugate partition; that is to say, one whose graph reads the same by columns as by rows.



Obviously we can dissect it into a Durfee square and two perfectly similar appended graphs, lateral and subjacent respectively. If the square involve m^2 nodes, the lateral graph may have any number of columns but not more than m rows; similarly the similar subjacent graph may have any number of rows but not more than m columns.

The lateral graph may be appended in as many ways as the number $\frac{1}{2}(n - m^2)$ possesses partitions into not more than m parts.

We are therefore concerned with the coefficient of

$$x^{\frac{1}{2}(n-m^2)}$$

in the development of the fraction

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^m)},$$

or of x^n in the development of

$$\frac{x^{m^2}}{(1-x^2)(1-x^4)\dots(1-x^{2m})}.$$

It follows, giving m all possible values, that the function which enumerates self-conjugate partitions can be expressed by the series

$$1 + \frac{x}{1-x^2} + \frac{x^4}{(1-x^2)(1-x^4)} + \frac{x^9}{(1-x^2)(1-x^4)(1-x^6)} + \dots$$
$$+ \frac{x^{m^2}}{(1-x^2)(1-x^4)\dots(1-x^{2m})} + \dots.$$

252. Another expression is obtainable from the circumstance that the symmetrical graphs which appertain to these partitions can be dissected in another manner, viz. into a succession of angles of nodes fitting into one another. Each such angle involves an uneven number of nodes, and no two angles involve the same number.

Thus in the graph depicted which denotes the self-conjugate partition 85543111 of the number 28, the successive angles are *ABC*, *DEF*, *GHI* and *J*, involving 15, 7, 5, 1 nodes respectively.

We find in fact that every symmetrical graph denotes a partition into unequal uneven numbers, and we are led to the identity

$$(1+x)(1+x^3)(1+x^5)\dots \text{ ad inf.}$$

$$=1+\frac{x}{1-x^2}+\frac{x^4}{(1-x^2)(1-x^4)}+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}+\dots$$

because the left-hand side infinite product enumerates the partitions into unequal uneven parts.

Moreover, if there be a square of m^2 nodes, the partition involves exactly m unequal uneven numbers; it follows that the coefficient of $a^m x^n$ i infinite product

$$(1+ax)(1+ax^3)(1+ax^5)\dots$$

is equal to the coefficient of x^n in the development of

$$\frac{x^{m^2}}{(1-x^2)(1-x^4)\dots(1-x^{2m})},$$

a fact which establishes the identity

$$(1+ax)(1+ax^3)(1+ax^5)\dots(1+ax^{2m-1})\dots \text{ ad inf.}$$

$$=1+a\frac{x}{1-x^2}+a^2\frac{x^4}{(1-x^2)(1-x^4)}+\dots+a^m\frac{x^{m^2}}{(1-x^2)(1-x^4)\dots(1-x^{2m})}+\dots.$$

This is a well-known result, but it was obtained in the above manner by Sylvester. It is a good example of the application of graph dissection to obtain results in algebraic series.

253. In a similar manner the reader will have no difficulty in dealing with the finite continued product and in establishing the formula

$$(1+ax)(1+ax^3)(1+ax^5)\dots(1+ax^{2i-1})$$

$$=1+a\frac{1-x^{2i}}{1-x^2}x+a^2\frac{(1-x^{2i-2})(1-x^{2i})}{(1-x^2)(1-x^4)}x^4+\dots$$

$$+a^m\frac{(1-x^{2i-2m+2})(1-x^{2i-2m+4})\dots(1-x^{2i})}{(1-x^2)(1-x^4)\dots(1-x^{2m})}x^{m^2}+\dots.$$

The number of self-conjugate partitions which involve exactly i parts may be similarly discussed. If a symmetrical graph involve a square of m^2 nodes the subjacent graph will involve exactly $i-m$ rows and not more than m columns; it will denote a partition of the number $\frac{1}{2}(n-m^2)$ into exactly $i-m$ parts, no part exceeding m in magnitude. The subjacent graphs will therefore be enumerated by the coefficient of $a^{i-m}x^{\frac{1}{2}(n-m^2)}$ in the expansion of

$$\frac{1}{(1-ax)(1-ax^2)\dots(1-ax^m)},$$

or by the coefficient of $x^{\frac{1}{2}(n-m^2)}$ in

$$x^{i-m}\frac{(1-x^m)(1-x^{m+1})\dots(1-x^{i-1})}{(1-x)(1-x^2)\dots(1-x^{i-m})},$$

or by the coefficient of x^{n-m^2} in

$$x^{2i-2m}\frac{(1-x^{2m})(1-x^{2m+2})\dots(1-x^{2i-2})}{(1-x^2)(1-x^4)\dots(1-x^{2i-2m})},$$

or by the coefficient of x^n in

$$x^{m^2-2m+2i}\frac{(1-x^{2m})(1-x^{2m+2})\dots(1-x^{2i-2})}{(1-x^2)(1-x^4)\dots(1-x^{2i-2m})};$$

and now giving m the values 1, 2, 3, ... in succession the enumerating function is found to be

$$x^{2i-1}+x^{2i}\frac{1-x^{2i-2}}{1-x^2}+x^{2i+3}\frac{(1-x^{2i-4})(1-x^{2i-2})}{(1-x^2)(1-x^4)}$$
$$+x^{2i+8}\frac{(1-x^{2i-6})(1-x^{2i-4})(1-x^{2i-2})}{(1-x^2)(1-x^4)(1-x^6)}+\dots.$$

Corresponding to these partitions we find from the dissection of the graphs into angles that we have partitions into unequal uneven parts which involve a highest part exactly equal to $2i-1$.

The number of these partitions is equal to the coefficient of x^n in the finite product

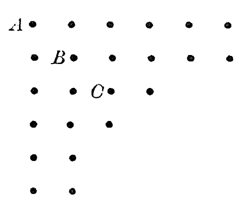
$$x^{2i-1}(1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3}),$$

that is to say, in

$$x^{2i-1}\left\{1+\frac{1-x^{2i-2}}{1-x^2}x+\frac{(1-x^{2i-4})(1-x^{2i-2})}{(1-x^2)(1-x^4)}x^4\right.$$
$$\left.+\frac{(1-x^{2i-6})(1-x^{2i-4})(1-x^{2i-2})}{(1-x^2)(1-x^4)(1-x^6)}x^9+\dots\right\},$$

a verification.

254. There is another way of investigating the generating function of conjugate partitions which leads to its expression by another series.



Consider this graph of a self-conjugate partition 664322.

It is symmetrical about the diagonal ABC, to the right of which we fin an *irregular* graph of three rows, viz.:

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & & \end{array}$$

If there are s nodes in the diagonal the irregular graph und have either $s-1$ or s rows. If we transform the irregular grap the second, third, etc. rows to the left so as to bring the left-l those rows underneath the left-hand node of the first row, w *regular* graph

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & \end{array}$$

which has the property that *no two rows can be of the same length.* Now the function which enumerates such graphs is either the product

$$(1+x)(1+x^2)(1+x^3)+\ldots,$$

or the series (which is algebraically equal to it)

$$1+\frac{x}{\mathbf{(1)}}+\frac{x^3}{\mathbf{(1)\,(2)}}+\frac{x^6}{\mathbf{(1)\,(2)\,(3)}}+\ldots+\frac{x^{\binom{s+1}{2}}}{\mathbf{(1)\,(2)}\ldots\mathbf{(s)}}+\ldots.$$

In the series the term

$$\frac{x^{\binom{s+1}{2}}}{\mathbf{(1)\,(2)}\ldots\mathbf{(s)}}$$

enumerates the partitions into exactly s parts.

Suppose now that the irregular graph, and therefore the derived regular graph also, has s rows exactly. Then the diagonal of nodes $ABC\ldots$ may have s or $s+1$ nodes. There is a corresponding graph consisting of s columns below the diagonal of nodes. These are also enumerated by

$$\frac{x^{\binom{s+1}{2}}}{\mathbf{(1)\,(2)}\ldots\mathbf{(s)}}.$$

Hence considering the combined contents of the graphs to the right of and below the diagonal of nodes we see that the enumerating function is found by writing x^2 for x in the above function. It is

$$\frac{x^{s(s+1)}}{\mathbf{(2)\,(4)}\ldots\mathbf{(2s)}}.$$

That is to say that the number w of nodes can be arranged in a pair of irregular graphs, competent to be the right-hand and lower graphs of

a self-conjugate partition, in a number of ways given by the coefficient of x^w in

$$\frac{x^{s(s+1)}}{(2)(4)\ldots(2s)},$$

the right-hand graph having s rows.

Containing only the two graphs the self-conjugate graph is so far invertebrate. It has no diagonal of nodes. Since this diagonal may have s or $s+1$ nodes, we must multiply by

$$x^s + x^{s+1} \equiv x^s \frac{(2)}{(1)},$$

to make the graph complete.

Hence as regards exactly s rows in the right-hand irregular graph, we have the enumerating function

$$x^s \frac{(2)}{(1)} \cdot \frac{x^{s(s+1)}}{(2)(4)\ldots(2s)}.$$

Hence the generating function for self-conjugate partitions is

$$\sum_0^\infty x^s \frac{(2)}{(1)} \cdot \frac{x^{s(s+1)}}{(2)(4)\ldots(2s)}$$

$$= 1 + x + \frac{x^3}{(1)} + \frac{x^8}{(1)(4)} + \frac{x^{15}}{(1)(4)(6)} + \ldots + \frac{x^{(s+1)^2-1}}{(1)(4)(6)\ldots(2s)} + \ldots,$$

and the term $\qquad \dfrac{x^{(s+1)^2-1}}{(1)(4)(6)\ldots(2s)}$

enumerates those of them which have s rows in that portion of the graph which lies to the right of the diagonal of nodes. The series obtained is therefore equal to the product previously found

$$(1+x)(1+x^3)(1+x^5)\ldots \text{ ad inf.}$$

and to the series

$$1 + \frac{x}{(2)} + \frac{x^4}{(2)(4)} + \frac{x^9}{(2)(4)(6)} + \ldots + \frac{x^{s^2}}{(2)(4)\ldots(2s)} + \ldots.$$

255. We are also led to the identity (by dividing each side of the identity which involves the old product and the new series by $1+x$)

$$(1+x^3)(1+x^5)(1+x^7)\ldots \text{ ad inf.}$$

$$= 1 + \frac{x^3}{(2)} + \frac{x^8}{(2)(4)} + \frac{x^{15}}{(2)(4)(6)} + \ldots + \frac{x^{(s+1)^2-1}}{(2)(4)\ldots(2s)} + \ldots.$$

The truth of this result can be seen from *à priori* considerations. For since $(s+1)^2-1$ is equal to s terms of the series

$$3 + 5 + 7 + 9 + \ldots$$

we can write down the graph of the partition

$$(2s+1,\ 2s-1,\ \dots\ 5,\ 3)$$

and join it to the graph which denotes a partition into s or fewer even numbers and thus obtain a graph which denotes a partition into s uneven unequal parts, no one of which is less than three. Such partitions are therefore enumerated by the $(s+1)$th term of the above series. The whole series is therefore equal to the product concerned, which obviously enumerates all partitions into uneven unequal parts, no one of which is less than three.

256. The general term in the expansion of the infinite continued product

$$(1-x)(1-x^2)(1-x^3)\ \dots \text{ ad inf.}$$

was obtained by Euler and is a celebrated result.

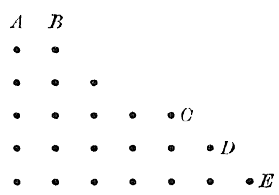
He shewed that the expansion is

$$\sum_{j=-\infty}^{j=+\infty} (-)^j x^{\frac{3j^2+j}{2}}.$$

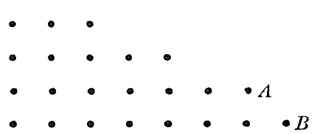
The coefficient of x^n in the expansion denotes the excess of the number of partitions of n into an even number of unequal parts over the number into an uneven number of unequal parts.

Many proofs of the theorem exist. One of the most interesting is that given by Franklin in the *Comptes Rendus* of the Institut (1880).

Suppose a partition into unequal parts written in ascending order of magnitude. Its graph consists of rows of nodes which continually increase in length. Consider the particular graph



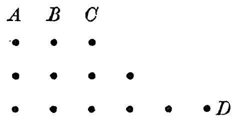
of the partition 23567. In particular fix the attention upon the nodes A, B at the *summit* (of the graph) and upon the nodes C, D, E which form a *slope line at the base* (of the graph). We can remove the nodes A, B from the summit and place them so as to form a new slope line at the base in the manner



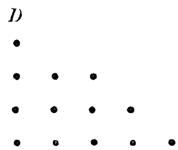
This is also the graph of a partition into unequal parts. The process, which has been termed "contraction," can be applied, with exceptions to be

specified later, whenever the nodes in the summit are equal to or less than the number of nodes in the slope line.

When this is not the case we can reverse the process, viz. remove the nodes in the base line and place them so as to form a new summit. For example, take the graph



which now becomes

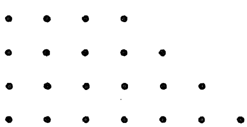


the process being termed "protraction." This also produces a partition into unequal parts.

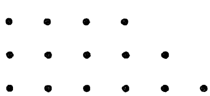
Subject to exceptional cases, when neither contraction nor protraction can be applied, contraction or protraction can be applied and the number of parts in the partition (always into unequal parts) is thereby changed from an even to an uneven number or *vice versâ*.

Hence, subject to elimination of the exceptional cases, there is a one-to-one correspondence between the partitions into an even number of parts and those into an uneven number.

The exceptional cases arise when the summit meets the slope line and contains either the same number or one more than the number of nodes in that line, for then neither contraction nor protraction is feasible. To explain this statement observe that to the graph



in which the summit meets the slope line and contains the same number of nodes, neither contraction nor protraction can be applied. Also that to the graph



contraction cannot be applied, for it produces the graph of a partition *with two parts equal*; protraction is obviously out of the question.

The exceptional cases therefore arise when the graphs correspond to one or other of the partitions

$$j, \quad j+1, \quad j+2, \ldots 2j-1,$$
$$j+1, \quad j+2, \quad j+3, \ldots \quad 2j,$$

where j may be unity or any larger number.

The contents of the two partitions are

$$\tfrac{1}{2}(3j^2-j), \quad \tfrac{1}{2}(3j^2+j) \text{ respectively,}$$

and each involves j parts; when j is even in each case there is no corresponding partition into an uneven number of parts obtainable by contraction or protraction, and therefore the number of partitions of $\frac{1}{2}(3j^2-j)$ or $\frac{1}{2}(3j^2+j)$ into an even number of unequal parts must exceed the number into an uneven number of unequal parts by unity. Further if j be uneven the number of partitions of $\frac{1}{2}(3j^2-j)$ or of $\frac{1}{2}(3j^2+j)$ into an uneven number of parts must be in excess by unity. These facts are all summed up in the identity

$$(1-x)(1-x^2)(1-x^3)\ldots \text{ad inf.} = \sum_{j=-\infty}^{j=+\infty} (-)^j x^{\frac{3j^2+j}{2}},$$

the numbers $\frac{1}{2}(3j^2-j)$ and $\frac{1}{2}(3j^2+j)$ being the direct and retrograde pentagonal numbers.

Jacobi's Identity.

257. An important theorem in this subject is derived by Jacobi from his researches in elliptic functions*, viz. it has the two statements

$$\begin{aligned}(1+x^{n-m})(1+x^{n+m})(1-x^{2n})&.(1+x^{3n-m})(1+x^{3n+m})(1-x^{4n})\\ &\times(1+x^{5n-m})(1+x^{5n+m})(1-x^{6n})\ldots\\ &= \sum_{-\infty}^{+\infty} x^{ni^2+mi},\end{aligned}$$

$$\begin{aligned}(1-x^{n-m})(1-x^{n+m})(1-x^{2n})&.(1-x^{3n-m})(1-x^{3n+m})(1-x^{4n})\\ &\times(1-x^{5n-m})(1-x^{5n+m})(1-x^{6n})\ldots\\ &= \sum_{-\infty}^{+\infty} (-)^i x^{ni^2+mi},\end{aligned}$$

wherein n and m are not necessarily integers.

258. The important particular cases are:

(i) In the second formula put $n=\frac{3}{2}$, $m=\frac{1}{2}$, deriving Euler's formula discussed in the last Article.

* *Crelle*, Vol. XXXII. p. 166.

259. (ii) In the first formula put $n=\frac{1}{2}$, $m=\frac{1}{2}$, deriving a series given by Gauss*, viz.:

$$(1+x)(1+x^2)(1+x^3)\ldots\times(1-x^2)(1-x^4)(1-x^6)\ldots$$
$$=1+x+x^3+x^6+x^{10}+\ldots+x^{\frac{1}{2}(i^2+i)}+\ldots,$$

the left-hand side of which may be also written

$$\frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

260. (iii) In the second formula put $n=\frac{5}{2}$, $m=\frac{3}{2}$, obtaining

$$(1-x)(1-x^4)(1-x^5).(1-x^6)(1-x^9)(1-x^{10}).(1-x^{11})(1-x^{14})(1-x^{15})\ldots.$$
$$=1-(x+x^4)+(x^7+x^{13})-(x^{18}+x^{27})+\ldots(-)^i(x^{\frac{5}{2}i^2-\frac{3}{2}i}+x^{\frac{5}{2}i^2+\frac{3}{2}i})\ldots,$$

the exponents on the left-hand side being numbers of the form

$$0,\ 1,\ 4 \bmod 5.$$

261. (iv) In the second formula put $n=\frac{5}{2}$, $m=\frac{1}{2}$, obtaining

$$(1-x^2)(1-x^3)(1-x^5).(1-x^7)(1-x^8)(1-x^{10}).(1-x^{12})(1-x^{13})(1-x^{15})\ldots.$$
$$=1-(x^2+x^3)+(x^9+x^{11})-(x^{21}+x^{24})+\ldots(-)^i(x^{\frac{5}{2}i^2-\frac{1}{2}i}+x^{\frac{5}{2}i^2+\frac{1}{2}i})\ldots,$$

the exponents on the left-hand side being numbers of the form

$$0,\ 2,\ 3 \bmod 5.$$

262. (v) It has been shewn by Sylvester† that, taking the second formula and making $n=\frac{1}{2}$, $m=\frac{1}{2}+\epsilon$ where ϵ is an infinitesimal, another formula of Jacobi may be deduced, viz.:

$$(1-x)^3(1-x^2)^3(1-x^3)^3\ldots=1-3x+5x^3-7x^6+\ldots(-)^i(2i+1)x^{\binom{i+1}{2}}\ldots.$$

This celebrated result is justly considered to be one of the most remarkable in the whole range of pure mathematics.

263. There is an important expansion of

$$\frac{1}{(1-ax)(1-a^2x)(1-a^3x)\ldots\text{ ad inf.}},$$

for if we suppose that it can be expanded in the form

$$1+A\frac{x}{1-ax}+B\frac{x^2}{(1-ax)(1-a^2x)}+C\frac{x^3}{(1-ax)(1-a^2x)(1-a^3x)}+\ldots,$$

where $A, B, C, \ldots$ etc. do not involve x, we substitute ax for x. This has the effect of multiplying the algebraic fraction by $1-ax$ while the series becomes

$$1+A\frac{ax}{1-a^2x}+B\frac{a^2x^2}{(1-a^2x)(1-a^3x)}+C\frac{a^3x^3}{(1-a^2x)(1-a^3x)(1-a^4x)}+\ldots.$$

* Göttingen Commentaries for the years 1808–11.

† *Collected Mathematical Papers*, Vol. IV. p. 60.

Hence this series must be equal to

$$(1-ax)\left\{1+A\frac{x}{1-ax}+B\frac{x^2}{(1-ax)(1-a^2x)}\right.$$
$$\left.+C\frac{x^3}{(1-ax)(1-a^2x)(1-a^3x)}+\dots\right\}.$$

Comparing these two series in regard to powers of x we find without difficulty

$$A=\frac{a}{1-a},$$
$$B=\frac{a^4}{(1-a)(1-a^2)},$$
$$C=\frac{a^9}{(1-a)(1-a^2)(1-a^3)},$$
$$\dots\dots\dots\dots\dots\dots$$

and thence the formula

$$\frac{1}{(1-ax)(1-a^2x)(1-a^3x)\dots\text{ ad inf.}}$$
$$=1+\frac{a}{1-a}\cdot\frac{x}{1-ax}+\frac{a^4}{(1-a)(1-a^2)}\cdot\frac{x^2}{(1-ax)(1-a^2x)}$$
$$+\frac{a^9}{(1-a)(1-a^2)(1-a^3)}\cdot\frac{x^3}{(1-ax)(1-a^2x)(1-a^3x)}+\dots.$$

264. Putting herein $x=1$ and $a=x$ we have

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\dots\text{ ad inf.}}$$
$$=1+\frac{x}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+\frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\dots,$$

a new form of the function which enumerates the unrestricted partitions of numbers.

The expansion can be at once interpreted by means of the Durfee square of nodes. The $(i+1)$th term of the series enumerates all the partitions whose graphs involve a Durfee square of i^2 nodes. For the lateral appendage to the square may be the graph of any partition of any number into not more than i parts, the part magnitude being unrestricted; and the subjacent appendage may be the graph of any partition of any number where the part magnitude is restricted not to exceed i, and the number of parts is unrestricted. Each appendage may involve partitions enumerated by

$$\frac{1}{(1)(2)\dots(i)}.$$

Hence the whole of the partitions which involve a Durfee square of i^2 nodes when graphically considered are enumerated by

$$\frac{x^{i^2}}{\{(1)(2)(3)\dots(i)\}^2},$$

and therefore the whole of the partitions of all numbers by

$$\sum_i \frac{x^{i^2}}{\{(1)(2)(3)\dots(i)\}^2},$$

which is the theorem in question.

265. Similarly it is easy to find the corresponding theorem when the number of denominator factors is limited. Thus

$$\frac{1}{(1-ax)(1-a^2x)(1-a^3x)\dots(1-a^ix)}$$

$$=1+a\frac{1-a^i}{1-a}\cdot\frac{x}{1-ax}+a^4\frac{(1-a^{i-1})(1-a^i)}{(1-a)(1-a^2)}\cdot\frac{x^2}{(1-ax)(1-a^2x)}+\dots,$$

a series possessing $i+1$ terms.

Putting herein $x=1$, $a=x$, we find

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^i)}$$

$$=1+\frac{x(1-x^i)}{(1-x)^2}+\frac{x^4(1-x^{i-1})(1-x^i)}{(1-x)^2(1-x^2)^2}+\frac{x^9(1-x^{i-2})(1-x^{i-1})(1-x^i)}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\dots,$$

and the reader will have no difficulty in observing that the $(s+1)$th term enumerates partitions whose graphs involve a Durfee square of s^2 nodes, and whose part magnitude is limited by the number i.

The Transformation of certain Series.

266. We consider a square lattice of nodes where there is no limit to the number of nodes in the side of the square.

At these nodes suppose numbers to be placed in such wise that they are in arithmetical progression alike in each row and in each column. Such an arrangement involves in general four parameters, and has the representation

$$\begin{array}{ccccc}
r & r+s & r+2s & r+3s & \dots \\
r+t & r+s+t+u & r+2s+t+2u & r+3s+t+3u & \dots \\
r+2t & r+s+2t+2u & r+2s+2t+4u & r+3s+2t+6u & \dots \\
r+3t & r+s+3t+3u & r+2s+3t+6u & r+3s+3t+9u & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

the common differences being

$$s,\quad s+u,\quad s+2u,\quad s+3u,\dots$$

in the successive rows, and

$$t, \quad t+u, \quad t+2u, \quad t+3u, \ldots$$

in the successive columns respectively.

Let X_{pq} be a function of x and μ_{pq} a numerical coefficient; we then construct the tableau

$$\begin{array}{lllll}
\mu_{11}X_{11}^{r} & \mu_{12}X_{12}^{r+s} & \mu_{13}X_{13}^{r+2s} & \mu_{14}X_{14}^{r+3s} & \ldots \\
\mu_{21}X_{21}^{r+t} & \mu_{22}X_{22}^{r+s+t+u} & \mu_{23}X_{23}^{r+2s+t+2u} & \mu_{24}X_{24}^{r+3s+t+3u} & \ldots \\
\mu_{31}X_{31}^{r+2t} & \mu_{32}X_{32}^{r+s+2t+2u} & \mu_{33}X_{33}^{r+2s+2t+4u} & \mu_{34}X_{34}^{r+3s+2t+6u} & \ldots \\
\mu_{41}X_{41}^{r+3t} & \mu_{42}X_{42}^{r+s+3t+3u} & \mu_{43}X_{43}^{r+2s+3t+6u} & \mu_{44}X_{44}^{r+3s+3t+9u} & \ldots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

The transformation which we consider is concerned with the sum of this infinite number of functions and depends upon the circumstance that in certain cases the summation may be performed in a variety of different ways. This arises because the exponents in each row and in each column are in arithmetical progression, and in consequence the functions $X_{pq}^{r+\cdots}$ in each row and in each column are in geometrical progression.

267. There are four methods of summation which are principally effective:

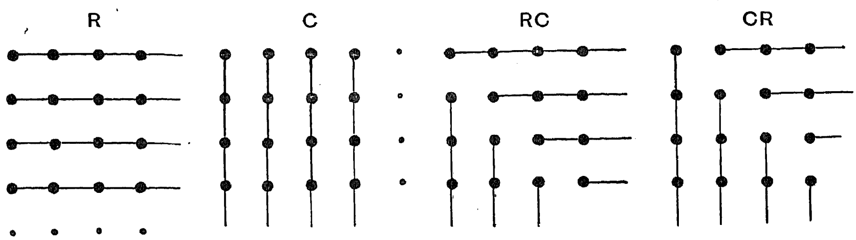
(i) We may be able to sum the functions in each row of the tableau.

(ii) We may be able to sum the functions in each column.

(iii) We may be able to sum, firstly, the functions in the first row; secondly, the remainder of the functions in the first column; thirdly, the remainder of those in the second row; fourthly, the remainder of those in the second column, and so forth.

(iv) We may sum, firstly, the functions in the first column; secondly, the remainder of those in the first row; thirdly, the remainder of those in the second column; fourthly, the remainder of those in the second row, and so forth.

Denote these modes of summation by R, C, RC and CR respectively, so that geometrically



As one of the simplest cases put

$$(r, s, t, u) = (1, 1, 2, 2),$$

$$X_{pq} = x, \quad \mu_{pq} = 1;$$

the tableau is

$$\begin{array}{ccccc} x & x^2 & x^3 & x^4 & \dots \\ x^3 & x^6 & x^9 & x^{12} & \dots \\ x^5 & x^{10} & x^{15} & x^{20} & \dots \\ x^7 & x^{14} & x^{21} & x^{28} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

and the summations R, C, RC, CR lead to the identically equal series

$$\frac{x}{1-x}+\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}+\frac{x^7}{1-x^7}+\dots,$$

$$\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^3}{1-x^6}+\frac{x^4}{1-x^8}+\dots,$$

$$\frac{x}{1-x}+\frac{x^3}{1-x^2}+\frac{x^6}{1-x^3}+\frac{x^{10}}{1-x^4}+\dots,$$

$$\frac{x}{1-x^2}+\frac{x^2}{1-x}+\frac{x^6}{1-x^4}+\frac{x^9}{1-x^3}+\dots,$$

or, as they may be written,

$$\sum_1 \frac{x^{2m-1}}{1-x^{2m-1}}=\sum_1 \frac{x^m}{1-x^{2m}}=\sum_1 \frac{x^{\binom{m+1}{2}}}{1-x^m}$$

$$=\sum_1 \frac{x^{m(2m-1)}}{1-x^{2m}}+\sum_1 \frac{x^{(m+1)(2m-1)}}{1-x^{2m-1}}.$$

268. The identity between the first and third of these series is interesting in the theory of partitions. The first clearly enumerates the number of ways of partitioning a number into parts which are repetitions of a single uneven number. The general term of the third may be written

$$x^{1+2+\dots+m}+x^{2+3+\dots+m+1}+x^{3+4+\dots+m+2}+\dots$$

and we gather that the series enumerates the ways of partitioning a number into parts which *form a single sequence.* We have thus the simplest case of the theorem which states that a number may be partitioned into parts which are repetitions of i distinct uneven numbers in as many ways as it may be partitioned into parts which form i distinct sequences. We have proved the theorem for $i=1$.

It will be observed that the first series also enumerates the uneven divisors of numbers.

269. In a similar manner it is easy to establish Jacobi's formula

$$\frac{x}{1+x}-\frac{x^3}{1+x^3}+\frac{x^5}{1+x^5}-\frac{x^7}{1+x^7}+\dots$$

$$=\frac{x}{1+x}-\frac{x^3}{1+x^2}+\frac{x^6}{1+x^3}-\frac{x^{10}}{1+x^4}+\dots.$$

For this purpose we put *as before*

$$(r, s, t, u) = (1, 1, 2, 2),$$
$$X_{pq} = x, \text{ but } \mu_{pq} = (-)^{p+q};$$

the tableau is then

$$\begin{array}{cccc}
x & -x^2 & x^3 & -x^4 \ldots \\
-x^3 & x^6 & -x^9 & x^{12} \ldots \\
x^5 & -x^{10} & x^{15} & -x^{20} \ldots \\
-x^7 & x^{14} & -x^{21} & x^{28} \ldots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

and we find that the summation R gives

$$\frac{x}{1+x} - \frac{x^3}{1+x^3} + \frac{x^5}{1+x^5} - \frac{x^7}{1+x^7} + \ldots,$$

and the summation RC yields

$$\frac{x}{1+x} - \frac{x^3}{1+x^2} + \frac{x^6}{1+x^3} - \frac{x^{10}}{1+x^4} + \ldots,$$

establishing the formula.

Observe that the summations C and CR give respectively

$$\frac{x}{1+x^2} - \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} - \frac{x^4}{1+x^8} + \ldots,$$

$$\frac{x}{1+x^2} - \frac{x^2}{1+x} + \frac{x^6}{1+x^4} - \frac{x^9}{1+x^3} + \ldots.$$

270. If in the general tableau we put

$$X_{pq} = x, \quad \mu_{pq} = 1$$

we find that the modes of summation give respectively the series

$$\frac{x^r}{1-x^s} + \frac{x^{r+t}}{1-x^{s+u}} + \frac{x^{r+2t}}{1-x^{s+2u}} + \frac{x^{r+3t}}{1-x^{s+3u}} + \ldots,$$

$$\frac{x^r}{1-x^t} + \frac{x^{r+s}}{1-x^{t+u}} + \frac{x^{r+2s}}{1-x^{t+2u}} + \frac{x^{r+3s}}{1-x^{t+3u}} + \ldots,$$

$$\frac{x^r}{1-x^s} + \frac{x^{r+t}}{1-x^t} + \frac{x^{r+s+t+u}}{1-x^{s+u}} + \frac{x^{r+s+2t+2u}}{1-x^{t+u}}$$
$$+ \frac{x^{r+2s+2t+4u}}{1-x^{s+2u}} + \frac{x^{r+2s+3t+6u}}{1-x^{t+2u}} + \ldots,$$

$$\frac{x^r}{1-x^t} + \frac{x^{r+s}}{1-x^s} + \frac{x^{r+s+t+u}}{1-x^{t+u}} + \frac{x^{r+2s+t+2u}}{1-x^{s+u}}$$
$$+ \frac{x^{r+2s+2t+4u}}{1-x^{t+2u}} + \frac{x^{r+3s+2t+6u}}{1-x^{s+2u}} + \ldots,$$

so that they are equal to one another. If we take $\mu_{pq}=(-)^{p+q}$ the first series becomes

$$\frac{x^r}{1+x^s}-\frac{x^{r+t}}{1+x^{s+u}}+\frac{x^{r+2t}}{1+x^{s+2u}}-\frac{x^{r+3t}}{1+x^{s+3u}}+\dots,$$

and the other three are *similarly* altered as to signs.

271. If we take
$$(r, s, t, u)=(1, 1, 2, 1)$$
$$X_{pq}=x \quad\text{and}\quad \mu_{pq}=pq,$$
we find the tableau

$$\begin{array}{cccc}
x & 2x^2 & 3x^3 & 4x^4 \dots \\
2x^3 & 4x^5 & 6x^7 & 8x^9 \dots \\
3x^5 & 6x^8 & 9x^{11} & 12x^{14} \dots \\
4x^7 & 8x^{11} & 12x^{15} & 16x^{19} \dots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

and the R and C summations give us the relation

$$\frac{x}{(1-x)^2}+\frac{2x^3}{(1-x^2)^2}+\frac{3x^5}{(1-x^3)^2}+\frac{4x^7}{(1-x^4)^2}+\dots$$
$$=\frac{x}{(1-x^2)^2}+\frac{2x^2}{(1-x^3)^2}+\frac{3x^3}{(1-x^4)^2}+\frac{4x^4}{(1-x^5)^2}+\dots.$$

Similarly, when $\mu_{pq}=\binom{p+1}{2}\binom{q+1}{2}$, we reach the identity

$$\frac{x}{(1-x)^3}+\frac{3x^3}{(1-x^2)^3}+\frac{6x^5}{(1-x^3)^3}+\frac{10x^7}{(1-x^4)^3}+\dots$$
$$=\frac{x}{(1-x^2)^3}+\frac{3x^2}{(1-x^3)^3}+\frac{6x^3}{(1-x^4)^3}+\frac{10x^4}{(1-x^5)^3}+\dots,$$

and in general when

$$\mu_{pq}=\binom{p+m-1}{m}\binom{q+m-1}{m},$$

we obtain

$$\frac{x}{(1-x)^{m+1}}+\binom{m+1}{1}\frac{x^3}{(1-x^2)^{m+1}}+\binom{m+2}{2}\frac{x^5}{(1-x^3)^{m+1}}$$
$$+\binom{m+3}{3}\frac{x^7}{(1-x^4)^{m+1}}+\dots$$
$$=\frac{x}{(1-x^2)^{m+1}}+\binom{m+1}{1}\frac{x^2}{(1-x^3)^{m+1}}+\binom{m+2}{2}\frac{x^3}{(1-x^4)^{m+1}}$$
$$+\binom{m+3}{3}\frac{x^4}{(1-x^5)^{m+1}}+\dots.$$

When
$$\mu_{pq}=(-)^{p+q}\binom{p+m-1}{m}\binom{q+m-1}{m},$$

the signs in all the denominators are changed and also the signs of alternate fractions on both sides.

272. Other identities given by Jacobi in the *Fundamenta Nova Theoriae Functionum Ellipticarum* are as follows:

$$(r, s, t, u) = (1, 2, 2, 4)$$

$$X_{pq} = x, \quad \mu_{pq} = (-)^{p+q}$$

gives by the summations R and C

$$\frac{x}{1-x^2} - \frac{x^3}{1-x^6} + \frac{x^5}{1-x^{10}} - \frac{x^7}{1-x^{14}} + \dots$$

$$= \frac{x}{1+x^2} + \frac{x^3}{1+x^6} + \frac{x^5}{1+x^{10}} + \frac{x^7}{1+x^{14}} + \dots.$$

This is the identity of Jacobi, but if we take the summations RC and CR we obtain two other forms of the series

$$\frac{x}{1-x^2} - \frac{x^3}{1+x^2} - \frac{x^9}{1-x^6} + \frac{x^{15}}{1+x^6} + \frac{x^{25}}{1-x^{10}} - \frac{x^{35}}{1+x^{10}} - \dots,$$

$$\frac{x}{1+x^2} + \frac{x^3}{1-x^2} - \frac{x^9}{1+x^6} - \frac{x^{15}}{1-x^6} + \frac{x^{25}}{1+x^{10}} + \frac{x^{35}}{1-x^{10}} - \dots.$$

273. Again if $$(r, s, t, u) = (1, 1, 1, 1)$$

$$X_{pq} = x,$$

and μ_{pq} numerically equal to p but with the negative sign, if both p and q are even, we find by the R and C summations

$$\frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \dots$$

$$= \frac{x}{(1-x)^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1-x^3)^2} + \frac{x^4}{(1+x^4)^2} + \dots$$

(see *loc. cit.* § 40).

274. Again if $$(r, s, t, u) = (1, 2, 2, 4)$$

$$X_{pq} = x, \quad \mu_{pq} = 2p - 1,$$

the R and C summations give

$$\frac{x}{1-x^2} + \frac{3x^3}{1-x^6} + \frac{5x^5}{1-x^{10}} + \frac{7x^7}{1-x^{14}} + \dots$$

$$= \frac{x(1+x^2)}{(1-x^2)^2} + \frac{x^3(1+x^6)}{(1-x^6)^2} + \frac{x^5(1+x^{10})}{(1-x^{10})^2} + \frac{x^7(1+x^{14})}{(1-x^{14})^2} + \dots$$

(see *loc. cit.* § 40).

275. Similarly
$$(r, s, t, u) = (1, 1, 2, 2)$$
$$X_{pq} = x, \quad \mu_{pq} = (-)^{p+q}(2p-1)^2$$
gives
$$\frac{x}{1+x} - \frac{9x^3}{1+x^3} + \frac{25x^5}{1+x^5} - \frac{49x^7}{1+x^7} + \dots$$
$$= \frac{x(1-6x^2+x^4)}{(1+x^2)^3} - \frac{x^2(1-6x^4+x^8)}{(1+x^4)^3} + \frac{x^3(1-6x^6+x^{12})}{(1+x^6)^3} - \dots,$$
and
$$(r, s, t, u) = (1, 2, 1, 2)$$
$$X_{pq} = x, \quad \mu_{pq} = p^3$$
gives
$$\frac{x}{1-x^2} + \frac{8x^2}{1-x^4} + \frac{27x^3}{1-x^6} + \frac{64x^4}{1-x^8} + \dots$$
$$= \frac{x(1+4x+x^2)}{(1-x)^4} + \frac{x^3(1+4x^3+x^6)}{(1-x^3)^4} + \frac{x^5(1+4x^5+x^{10})}{(1-x^5)^4} + \dots.$$

CHAPTER III

RAMANUJAN'S IDENTITIES

276. Mr Ramanujan of Trinity College, Cambridge, has suggested a large number of formulæ which have applications to the partition of numbers. Two of the most interesting of these concern partitions whose parts have a definite relation to the modulus five. Theorem I gives the relation

$$1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\dots$$
$$+\frac{x^{i^2}}{(1-x)(1-x^2)\dots(1-x^i)}+\dots$$
$$=\frac{1}{(1-x)(1-x^6)(1-x^{11})\dots(1-x^{5m+1})\dots}$$
$$\times\frac{1}{(1-x^4)(1-x^9)(1-x^{14})\dots(1-x^{5m+4})\dots},$$

where on the right-hand side the exponents of x are the numbers given by the congruences $\equiv 1 \bmod 5$, $\equiv 4 \bmod 5$.

This most remarkable theorem has been verified as far as the coefficient of x^{89} by actual expansion so that there is practically no reason to doubt its truth; but it has not yet been established.

The series on the left-hand side can be interpreted in the theory of partitions. In fact it has been discussed in a previous article (Art. 242). The circumstance that i^2 is equal to the sum of the first i uneven numbers shews that the algebraic fraction which has x^{i^2} for numerator enumerates the partitions of all numbers into exactly i parts between which there are neither repetitions nor sequences. Hence the whole series enumerates all partitions of any given number which involve neither repetitions nor sequences. The theorem asserts that the partitions, whose parts are limited to be of the forms $5m+1$, $5m+4$, are equi-numerous with those which involve neither repetitions nor sequences.

Thus the six-to-six correspondence for the partitions of ten is

Parts of the forms $5m \pm 1$	No repetitions or sequences
91	10
64	91
61^4	82
$4^2 1^2$	73
41^6	64
1^{10}	631

By a previous article the fraction $x^{i^2}/(1-x)(1-x^2) \ldots (1-x^i)$ also enumerates the partitions into exactly i parts, no part being less than i, so that the series enumerates partitions such that the smallest part is not less than the number of parts. These six partitions of the number ten are

$$10, \quad 82, \quad 73, \quad 64, \quad 55, \quad 433.$$

277. The right-hand side of the identity can be put into the form of the product of two series because, starting with the known expansion

$$\frac{1}{(1-a)(1-ax)(1-ax^2) \ldots \text{ad inf.}}$$
$$= 1 + \frac{a}{1-x} + \frac{a^2}{(1-x)(1-x^2)} + \frac{a^3}{(1-x)(1-x^2)(1-x^3)} + \ldots,$$

we put $a = x$ and $x = x^5$ simultaneously and find

$$\frac{1}{(1-x)(1-x^6)(1-x^{11})(1-x^{16}) \ldots}$$
$$= 1 + \frac{x}{(1-x^5)} + \frac{x^2}{(1-x^5)(1-x^{10})} + \frac{x^3}{(1-x^5)(1-x^{10})(1-x^{15})} + \ldots.$$

Similarly if we put $a = x^4$, $x = x^5$ simultaneously we find

$$\frac{1}{(1-x^4)(1-x^9)(1-x^{14})(1-x^{19}) \ldots}$$
$$= 1 + \frac{x^4}{(1-x^5)} + \frac{x^8}{(1-x^5)(1-x^{10})} + \frac{x^{12}}{(1-x^5)(1-x^{10})(1-x^{15})} + \ldots.$$

We can see almost intuitively that these results are true; take for instance the fraction $x^3/(1-x^5)(1-x^{10})(1-x^{15})$. We can realise the partitions which it enumerates by taking a vertical column of three nodes

$$\begin{array}{c}\bullet\\ \bullet\\ \bullet\end{array}$$

and placing to the right of it the nodal graph of any partition into three or fewer parts of the form $5m$. Hence the partitions enumerated have exactly three parts, each part of the form $5m+1$.

Similarly the fraction $x^{12}/(1-x^5)(1-x^{10})(1-x^{15})$ enumerates the partitions having exactly three parts, each part of the form $5m+4$.

Another statement of Ramanujan's First Theorem is therefore

$$1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\dots$$
$$=\left\{1+\frac{x}{1-x^5}+\frac{x^2}{(1-x^5)(1-x^{10})}+\dots\right\}\left\{1+\frac{x^4}{1-x^5}+\frac{x^8}{(1-x^5)(1-x^{10})}+\dots\right\}.$$

278. Theorem II gives the relation

$$1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\dots$$
$$+\frac{x^{i^2+i}}{(1-x)(1-x^2)\dots(1-x^i)}+\dots$$
$$=\frac{1}{(1-x^2)(1-x^7)(1-x^{12})\dots(1-x^{5m+2})\dots}$$
$$\times\frac{1}{(1-x^3)(1-x^8)(1-x^{13})\dots(1-x^{5m+3})\dots},$$

the exponents of x on the right-hand side being of the form $5m+2$ and $5m+3$.

This relation has also been verified by actual expansion to a high power of x, but it has not been established.

To interpret the series on the left-hand side we observe that the number i^2+i is the sum of the first i even numbers. In particular when $i=3$, since $12=6+4+2$ we form the graph of the partition 642 of the number twelve

$$\begin{array}{cccccc} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&&\\ \bullet&\bullet&&&& \end{array}$$

and find that if we place to the right of it the graph of any partition which has three parts the combined graph clearly denotes a partition which has exactly three parts, no part less than two, and which involves neither repetitions nor sequences of parts.

Hence we at once infer that the whole series enumerates partitions of all numbers which are such that no part is less than two and which involve neither sequences nor repetitions of parts.

Ramanujan's Second Theorem thus states that of any given number such partitions are equi-numerous with those whose parts are all of the forms $5m+2$, $5m+3$. Thus for the number ten the four-to-four correspondence is

Parts of the forms $5m\pm2$	No repetitions or sequences and no part <2
82	10
73	82
$3^2 2^2$	73

279. The series on the left-hand side has another interpretation due to the circumstance that i^2+i is the product of the numbers $i+1$, i.

Thus when $i=3$, since $12=4+4+4$ we take the graph of the partition 444 of the number twelve,

$$\begin{array}{cccc}\bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet\end{array}$$

and place to the right of it the graph of any partition which has three parts; the combined graph denotes a partition into exactly three parts and no part less than four. In general the combined graph denotes a partition into exactly i parts and no part less than $i+1$.

Thus the whole series enumerates partitions such that no part is less than the number which exceeds the number of parts by unity.

For the number ten these partitions are

$$10, \quad 73, \quad 64, \quad 55.$$

280. We may also obviously write Ramanujan's Second Theorem in the form

$$1+\frac{x^2}{(1-x)}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\ldots$$

$$=\left\{1+\frac{x^2}{(1-x^5)}+\frac{x^4}{(1-x^5)(1-x^{10})}+\frac{x^6}{(1-x^5)(1-x^{10})(1-x^{15})}+\ldots\right\}$$

$$\times\left\{1+\frac{x^3}{(1-x^5)}+\frac{x^6}{(1-x^5)(1-x^{10})}+\frac{x^9}{(1-x^5)(1-x^{10})(1-x^{15})}+\ldots\right\}.$$

Ramanujan himself deduces other results, of great interest and beauty, from these, but they are not given here as they are not directly connected with the combinatory analysis.

The circumstance that the two theorems given still await demonstration adds greatly to their interest for the student and stamps them as being very remarkable.

281. There is a correspondence between the whole of the partitions of a number and those particular partitions which involve neither repetitions nor sequences of parts.

If we consider the graph of any partition of any number, say the graph

$$\begin{array}{cccccc} & & & & & A \\ B\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & E\bullet & \bullet & \bullet\, D & & \\ \bullet & \bullet & H\bullet & \bullet\, G & & \\ \bullet & F\bullet & & & & \end{array}$$

of the partition $64^2 21$ of the number seventeen, and read the graph by enumerating the nodes in the angles

$$ABC,\ DEF,\ GH,$$

we obtain the partition 10, 5, 2 of the same number which *necessarily* has neither sequences nor repetitions. But this is not a one-to-one correspondence because several other partitions are made up of angles which, looking only to the numbers of nodes which they contain, are precisely the same and lead to the same partition 10, 5, 2.

Ex. gr. the partition $3^4 21^3$ having the graph

$$\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\bullet & & \\
\bullet & & \\
\bullet & &
\end{array}$$

leads by angle-reading to the same partition 10, 5, 2.

There is thus a group of partitions each member of which leads to the same partition which possesses neither repetitions nor sequences. Further, corresponding to every partition which possesses neither repetitions nor sequences, there will be a group of partitions and there is a one-to-one correspondence between the groups and the special partitions.

The problem of enumerating the special partitions is the same as that of enumerating the groups, of the nature considered, into which the whole of the partitions may be separated.

For the number six there are three groups, the correspondence being

groups	$(6,\ 51,\ 41^2,\ 31^3,\ 21^4,\ 1^6)$,	$(42,\ 321,\ 2^2 1^2)$,	$(3^2,\ 2^3)$,
special partitions	6 ,	51 ,	42 .

It is, in fact, evident that of the number m the group which belongs to the special partition (m) comprises the m members

$$s1^{m-s},$$

where s has values $0, 2, 3, \dots m$.

It is not difficult also to establish that corresponding to the special partition $(m_1 m_2)$ the group has

$$(m_1 - m_2 - 1)\, m_2$$

members.

282. It will now be proved that corresponding to the special partition $(m_1 m_2 \ldots m_s)$ the group has

$$(m_1 - m_2 - 1)(m_2 - m_3 - 1) \ldots (m_{s-1} - m_s - 1) m_s$$

members.

This will be accomplished by comparing the series

$$1 + \frac{x}{(1)} + \frac{x^4}{(1)(2)} + \frac{x^9}{(1)(2)(3)} + \ldots,$$

the first series of Ramanujan which enumerates the partitions without both repetitions and sequences, hereafter termed special partitions, with the series

$$1 + \frac{x}{(1)^2} + \frac{x^4}{(1)^2 (2)^2} + \frac{x^9}{(1)^2 (2)^2 (3)^2} + \ldots,$$

which is one form of the function which enumerates the whole of the partitions.

The whole of the partitions, whose graphs involve precisely s angles, are enumerated by

$$\frac{x^{s^2}}{(1)^2 (2)^2 \ldots (s)^2} = \Sigma (q_1 + 1)(q_2 + 1) \ldots (q_s + 1)\, x^{q_1 + 2q_2 + \ldots + sq_s + s^2},$$

where under the sign of summation the numerical magnitudes

$$q_1, q_2, \ldots q_s$$

may, each of them, have all values from zero to infinity.

If we write herein

$$\begin{aligned} m_s &= q_s + 1, \\ m_{s-1} &= q_{s-1} + q_s + 3, \\ m_{s-2} &= q_{s-2} + q_{s-1} + q_s + 5, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ m_1 &= q_1 + q_2 + \ldots + q_s + 2s - 1, \end{aligned}$$

then $$m_1 + m_2 + \ldots + m_s = q_1 + 2q_2 + 3q_3 + \ldots + sq_s + s^2,$$

and $$(m_1 m_2 m_3 \ldots m_s)$$

is a special partition involving precisely s parts.

Also $$\begin{aligned} q_s + 1 &= m_s, \\ q_{s-1} + 1 &= m_{s-1} - m_s - 1, \\ q_{s-2} + 1 &= m_{s-2} - m_{s-1} - 1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ q_2 + 1 &= m_2 - m_3 - 1, \\ q_1 + 1 &= m_1 - m_2 - 1. \end{aligned}$$

Hence

$$\frac{x^{s^2}}{(1)^2 (2)^2 \ldots (s)^2} = \Sigma (m_1 - m_2 - 1)(m_2 - m_3 - 1) \ldots (m_{s-1} - m_s - 1) m_s\, x^{m_1 + m_2 + \ldots + m_s},$$

the summation being for every special partition which involves exactly s parts.

But from the first series of Ramanujan

$$\frac{x^{s^2}}{(1)(2)\ldots(s)} = \Sigma\, x^{m_1+m_2+\ldots+m_s}.$$

Hence to every special partition

$$(m_1 m_2 \ldots m_s)$$

corresponds a group of

$$(m_1 - m_2 - 1)(m_2 - m_3 - 1)\ldots(m_{s-1} - m_s - 1)\, m_s$$

partitions drawn from the whole of the partitions.

283. It is thus seen that the number of partitions of n is

$$\Sigma\,(m_1 - m_2 - 1)(m_2 - m_3 - 1)\ldots(m_{s-1} - m_s - 1)\, m_s,$$

the summation being for every special partition

$$(m_1 m_2 \ldots m_s)$$

of the number n.

284. If we realise the number m_1 physically by a chain of m_1 beads connected at equal intervals by threads, and so also the other numbers $m_2, m_3, \ldots m_s$, the s chains can be placed in angles in

$$(m_1 - m_2 - 1)(m_2 - m_3 - 1)\ldots(m_{s-1} - m_s - 1)\, m_s$$

different ways to form different regular graphs of the number n.

285. The special partitions enumerated by the second theorem of Ramanujan are included in those enumerated by the first theorem. The same group of partitions is in correspondence with the same special partition in each case. In the case of the second theorem, however, the whole of the unrestricted partitions are not involved because those partitions which give rise by angle reading to a part 1 are necessarily omitted. If a graph involves a square of s^2 nodes an angle will consist of a single node if the lateral graph has at most $s-1$ rows and the subjacent graph at most $s-1$ columns.

The graphs to be omitted are therefore enumerated by

$$\frac{x^{s^2}}{(1)^2(2)^2\ldots(s-1)^2}.$$

Hence giving s all values not less than unity the whole of the omitted graphs are enumerated by

$$x + \frac{x^4}{(1)^2} + \frac{x^9}{(1)^2(2)^2} + \frac{x^{16}}{(1)^2(2)^2(3)^2} + \ldots;$$

and subtracting this function from that which enumerates the unrestricted graphs, viz.

$$1+\frac{x}{(1)^2}+\frac{x^4}{(1)^2(2)^2}+\frac{x^9}{(1)^2(2)^2(3)^2}+\dots,$$

we reach the function

$$1+\sum_{s=1}^{s=\infty}\left\{\frac{x^{s(s+1)}}{(1)^2(2)^2\dots(s-1)^2(s)^2}+\frac{x^{s(s+1)}}{(1)^2(2)^2\dots(s)^2}\right\},$$

as that which enumerates the partitions which compose the groups associated with the special partitions of the second theorem.

286. The left-hand side of the identity which constitutes Ramanujan's First Theorem has been shewn to be the function which enumerates the partitions which are without repetitions and sequences of parts.

If we wish to restrict the part magnitude we may proceed in the following manner:—Consider the partition into three parts

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & & & & \end{array}$$

When the part magnitude is unrestricted we construct the graph of the partition 531 of the number 9 and join to it any partition into three parts. If however the part magnitude is restricted not to exceed m, the partitions so added must be such that the part magnitude does not exceed $m-5$. These are enumerated by the function

$$\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)},$$

and give rise to the term

$$x^9\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)}$$

in the sought function.

When the partitions have i parts, the part magnitude of the added partitions must have a part magnitude not exceeding $2i-1$ and we are similarly led to the term

$$x^{i^2}\frac{(m-2i+2)(m-2i+3)\dots(m-i+1)}{(1)(2)\dots(i)}.$$

Hence the function required is

$$1+x\frac{(m)}{(1)}+x^4\frac{(m-2)(m-1)}{(1)(2)}+x^9\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)}$$
$$+\dots \text{ to } \mu \text{ terms.}$$

The series evidently does not extend to infinity because when the part magnitude is restricted there is a limit to the content of the partitions, and

also to the number of parts. When m is even the partition, of the nature considered, which involves the greatest number of parts is $m, m-2, m-4, \ldots 2$, and this has $\frac{1}{2}m$ parts. When m is uneven the partition in question is $m, m-2, m-4, \ldots 1$ and this has $\frac{1}{2}(m+1)$ parts. Hence i has the limit $\frac{1}{2}m$, or $\frac{1}{2}(m+1)$ according as m is even or uneven, and therefore μ the number of terms to be taken of the series has the value

$$\tfrac{1}{2}(m+2), \text{ or } \tfrac{1}{2}(m+3)$$

according as m is even or uneven.

287. We may expand the general term of the series

$$x^{i^2}\frac{(m-2i+2)(m-2i+3)\ldots(m-i+1)}{(1)(2)\ldots(i)}$$

by means of the theorem

$$(1-x^p)(1-x^{p+1})\ldots(1-x^{p+q-1})$$
$$=1-x^p\frac{(1-x^q)}{(1-x)}+x^{2p+1}\frac{(1-x^{q-1})(1-x^q)}{(1-x)(1-x^2)}-x^{3p+3}\frac{(1-x^{q-2})(1-x^{q-1})(1-x^q)}{(1-x)(1-x^2)(1-x^3)}$$
$$+\ldots \text{ to } q+1 \text{ terms,}$$

and we are led to the identity

$$1+x\frac{(m)}{(1)}+x^4\frac{(m-2)(m-1)}{(1)(2)}+x^9\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)}+\ldots \text{ to } \mu \text{ terms}$$
$$=\quad 1+\frac{x}{(1)}+\frac{x^4}{(1)(2)}+\frac{x^9}{(1)(2)(3)}+\ldots \text{ to } \mu \text{ terms}$$
$$-\frac{x^{m+1}}{(1)}\left\{1+\frac{x}{(1)}+\frac{x^4}{(1)(2)}+\frac{x^9}{(1)(2)(3)}+\ldots \text{ to } \mu-1 \text{ terms}\right\}$$
$$+\frac{x^{2m+1}}{(1)(2)}\left\{1+\frac{x}{(1)}+\frac{x^4}{(1)(2)}+\frac{x^9}{(1)(2)(3)}+\ldots \text{ to } \mu-2 \text{ terms}\right\}$$
$$\cdots\cdots\cdots\cdots$$
$$+(-)^s\frac{x^{\frac{1}{2}s(2m-s+3)}}{(1)(2)\ldots(s)}\left\{1+\frac{x}{(1)}+\frac{x^4}{(1)(2)}+\frac{x^9}{(1)(2)(3)}+\ldots \text{ to } \mu-s \text{ terms}\right\}$$
$$+\ldots\ldots,$$

a series which involves μ rows of terms.

288. The same function is also that part of the expansion of the product

$$(1+x)(1+x^2)(1+x^3)\ldots(1+x^m)$$

which presents itself when from the product of consecutive factors

$$(1+x^s)(1+x^{s+1})$$

the term x^{2s+1} is deleted.

This deletion insures that the partitions, which necessarily involve no repetitions, do not contain any sequence of parts.

We denote such a product by

$$M(1+x)(1+x^2)(1+x^3)\dots(1+x^m).$$

289. In Ramanujan's Second Theorem regarded in connexion with the enumeration of partitions which possess no repetitions, no sequences and no part less than two, we similarly restrict the part magnitude. For partitions into three parts we take the graph of the partition (642) of the number twelve, viz.

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & & \end{array}$$

and add to it the graph of a partition which has three parts and no part greater than $m-6$. We thus obtain the graph of a partition of the desired character and a term

$$x^{12}\frac{(\mathrm{m}-5)(\mathrm{m}-4)(\mathrm{m}-3)}{(1)(2)(3)}$$

of the enumerating function.

In general when there are i parts the term of the function is

$$x^{i(i+1)}\frac{(\mathrm{m}-2\mathrm{i}+1)(\mathrm{m}-2\mathrm{i}+2)\dots(\mathrm{m}-\mathrm{i})}{(1)(2)\dots(\mathrm{i})},$$

leading to the enumerating function

$$1+x^2\frac{(\mathrm{m}-1)}{(1)}+x^6\frac{(\mathrm{m}-3)(\mathrm{m}-2)}{(1)(2)}+x^{12}\frac{(\mathrm{m}-5)(\mathrm{m}-4)(\mathrm{m}-3)}{(1)(2)(3)}$$
$$+\dots \text{ to } \nu \text{ terms},$$

where ν has the value $\frac{1}{2}(m+2)$ or $\frac{1}{2}(m+1)$ according as m is even or uneven.

This series can be put into the form

$$1+\frac{x^2}{(1)}+\frac{x^6}{(1)(2)}+\frac{x^{12}}{(1)(2)(3)}+\dots \text{ to } \nu \text{ terms}$$
$$-\frac{x^{m+1}}{(1)}\left\{1+\frac{x^2}{(1)}+\frac{x^6}{(1)(2)}+\frac{x^{12}}{(1)(2)(3)}+\dots \text{ to } \nu-1 \text{ terms}\right\}$$
$$+\frac{x^{2m+1}}{(1)(2)}\left\{1+\frac{x^2}{(1)}+\frac{x^6}{(1)(2)}+\frac{x^{12}}{(1)(2)(3)}+\dots \text{ to } \nu-2 \text{ terms}\right\}$$
$$-\dots\dots$$
$$+(-)^s\frac{x^{\frac{1}{2}s(2m-s+3)}}{(1)(2)\dots(\mathrm{s})}\left\{1+\frac{x^2}{(1)}+\frac{x^6}{(1)(2)}+\frac{x^{12}}{(1)(2)(3)}+\dots \text{ to } \nu-s \text{ terms}\right\}$$
$$+\dots\dots,$$

a series which involves ν rows of terms.

The same function is also that part of the expansion of the product

$$(1+x^2)(1+x^3)\dots(1+x^m)$$

which presents itself when, from the product of consecutive factors,

$$(1+x^s)(1+x^{s+1}),$$

the term x^{2s+1} is deleted.

290. Let $V_{s,m}$ denote the function which enumerates partitions without repetitions and sequences and having no part less than s nor greater than m. Then

$$V_{1,m}=1+x\frac{(m)}{(1)}+x^4\frac{(m-2)(m-1)}{(1)(2)}+x^9\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)}+\dots,$$

$$V_{2,m}=1+x^2\frac{(m-1)}{(1)}+x^6\frac{(m-3)(m-2)}{(1)(2)}+x^{12}\frac{(m-5)(m-4)(m-3)}{(1)(2)(3)}+\dots,$$

$$V_{3,m}=1+x^3\frac{(m-2)}{(1)}+x^8\frac{(m-4)(m-3)}{(1)(2)}+x^{15}\frac{(m-6)(m-5)(m-4)}{(1)(2)(3)}+\dots,$$

etc.,

and considering the expressions

$$M(1+x)(1+x^2)\dots(1+x^m),$$
$$M(1+x^2)(1+x^3)\dots(1+x^m),$$
$$M(1+x^3)(1+x^4)\dots(1+x^m),$$
$$\dots\dots\dots\dots\dots\dots$$

we see that

$$M(1+x)(1+x^2)\dots(1+x^m)$$
$$=M(1+x^2)(1+x^3)\dots(1+x^m)+xM(1+x^2)(1+x^3)\dots(1+x^m),$$

and moreover $xM(1+x^2)(1+x^3)\dots(1+x^m)$ is effectively equivalent to

$$xM(1+x^3)\dots(1+x^m),$$

that is $$V_{1,m}=V_{2,m}+xV_{3,m}$$

and in general $$V_{s,m}=V_{s+1,m}+x^sV_{s+2,m},$$

relations readily verified from the series expressions.

291. From this result it is manifest that we can express $V_{s,m}$ as a linear function of $V_{1,m}$, $V_{2,m}$. We obtain the series of results

$$xV_{3,m}=V_{1,m}-V_{2,m},$$
$$x^3V_{4,m}=-V_{1,m}+(1+x)V_{2,m},$$
$$x^6V_{5,m}=(1+x^3)V_{1,m}-(1+x+x^2)V_{2,m},$$

etc.,

and observing that the last written equation may be written

$$x^6V_{5,m}=V_{2,2}V_{1,m}-V_{1,2}V_{2,m},$$

it is natural to assume that, as far as the integer s,

$$x^{\binom{s-1}{2}} V_{s,m} = (-)^{s+1} (V_{2,s-3} V_{1,m} - V_{1,s-3} V_{2,m}),$$

with the object of proving that the formula holds when $s+1$ is written for s. We require the theorem

$$V_{s,m} = V_{s,m-1} + x^m V_{s,m-2}.$$

Writing $V_{s,m}$ in the form of the *conditioned* product

$$M(1+x^s)(1+x^{s+1}) \ldots (1+x^{m-1})(1+x^m)$$

$$\equiv M(1+x^s)(1+x^{s+1}) \ldots (1+x^{m-1}) + x^m M(1+x^s)(1+x^{s+1}) \ldots (1+x^{m-1})$$

$$\equiv M(1+x^s)(1+x^{s+1}) \ldots (1+x^{m-1}) + x^m M(1+x^s)(1+x^{s+1}) \ldots (1+x^{m-2}),$$

we see that $$V_{s,m} = V_{s,m-1} + x^m V_{s,m-2}.$$

Now $$x^{s-1} V_{s+1,m} = V_{s-1,m} - V_{s,m},$$

and substituting for $V_{s-1,m}$, $V_{s,m}$ the assumed expressions linear in $V_{1,m}$, $V_{2,m}$ we find on reduction

$$(-)^s x^{\binom{s}{2}} V_{s+1,m} = (V_{2,s-3} + x^{s-2} V_{2,s-4}) V_{1,m} - (V_{1,s-3} + x^{s-2} V_{1,s-4}) V_{2,m},$$

or $$x^{\binom{s}{2}} V_{s+1,m} = (-)^{s+2} (V_{2,s-2} V_{1,m} - V_{1,s-2} V_{2,m}).$$

Hence if the theorem be true for the integer s and lower integers it is equally true for the integer $s+1$. It is thus established by induction.

292. The theorem is better written in the form

$$x^{\binom{s+2}{2}} V_{s+3,m} = (-)^s (V_{2,s} V_{1,m} - V_{1,s} V_{2,m}),$$

and in particular, when $m = \infty$,

$$x^{\binom{s+2}{2}} V_{s+3,\infty} = (-)^s (V_{2,s} V_{1,\infty} - V_{1,s} V_{2,\infty}).$$

In utilizing the formulæ the particular cases

$$V_{m,m} = 1 + x^m, \quad V_{m+1,m} = 1,$$

must be noted and it further appears that in the formula we must suppose that

$$V_{m+2,m} = (-)^m, \quad V_{m+3,m} = 0.$$

The particular case

$$x^{\binom{m-1}{2}} V_{m,m} = (-)^{m+1} (V_{2,m-3} V_{1,m} - V_{1,m-3} V_{2,m})$$

is interesting since the left-hand side has a very simple expression. It may be verified in the simplest cases.

It will be noticed that the two Ramanujan series when modified so as to shew a maximum part magnitude are the only fundamental series in this branch of the theory of partitions.

293. The Generating Function $V_{3,\infty}$ which enumerates partitions without repetitions or sequences which have no part less than three has, by Ramanujan's results, the simple expression

$$\frac{1}{x}\left\{\frac{1}{(1)(4)(6)(9)(11)(14)\ldots}-\frac{1}{(2)(3)(7)(8)(12)(13)\ldots}\right\},$$

which, by previous theorems, can be thrown into the form

$$\frac{(1)(2)-x^6(2)(4)+x^{17}(3)(6)-x^{33}(4)(8)+x^{54}(5)(10)-\ldots}{(1)(2)(3)\ldots\text{ ad inf.}},$$

where the general numerator term is

$$(-)^{s+1}x^{\frac{1}{2}(s-1)(5s+2)}(s)(2s).$$

294. The two theorems

$$V_{s,m}=V_{s+1,m}+x^sV_{s+2,m},$$
$$V_{s,m}=V_{s,m-1}+x^mV_{s,m-2},$$

are complementary and arise from reasoning in regard to the beginning and end of the conditioned product respectively.

From them we derive the two series

$$V_{1,m}=V_{2,m}+xV_{4,m}+x^4V_{6,m}+x^9V_{8,m}+\ldots+x^{s^2}V_{(2s+2),m}+\ldots,$$
$$V_{1,m}=V_{1,m-1}+x^mV_{1,m-3}+x^{2m-2}V_{1,m-5}+x^{3m-6}V_{1,m-7}+\ldots+x^{(m+1)s-s^2}V_{1,m-2s-1}+\ldots,$$

and we can interpret each term in both series.

The partitions enumerated by the sth term in the first series are those which, while involving no repetitions or sequences, have

(i) no part equal to $2s-1$;

(ii) the parts $1, 3, 5, \ldots 2s-3$;

(iii) other parts equal to or greater than $2s$ and not greater than m.

Those enumerated by the sth term in the second series are those which, while involving neither repetitions nor sequences, have

(i) no part equal to $m-2s+2$;

(ii) the parts $m-2s+4,\ m-2s+6, \ldots m$;

(iii) other parts equal to or less than $m-2s+1$.

Complementary to the formula

$$x^{\binom{s+2}{2}}V_{s+3,m}=(-)^s(V_{2,s}V_{1,m}-V_{1,s}V_{2,m}),$$

we establish in the same manner the formula

$$x^{\frac{1}{2}(s-1)(2m-s+2)}V_{1,m-s}=(-)^s(V_{m-s+3,m-1}V_{1,m}-V_{m-s+3,m}V_{1,m-1}).$$

We can also obtain formulæ by reasoning in regard to a factor of the conditioned product which is not the first or last factor.

Thus since

$$V_{s,m} = M(1+x^s)\dots(1+x^{t-1})(1+x^{t+1})\dots(1+x^m)$$
$$+M(1+x^s)\dots(1+x^{t-2})x^t(1+x^{t+2})\dots(1+x^m),$$

we have $$V_{s,m} = V_{s,t-1}V_{t+1,m} + x^t V_{s,t-2}V_{t+2,m},$$

an inclusive formula; for if

(i) $t=s$, $$V_{s,m} = V_{s+1,m} + x^s V_{s+2,m},$$

(ii) $t=m$, $$V_{s,m} = V_{s,m-1} + x^m V_{s,m-2}.$$

295. It is a remarkable circumstance that the present theory is connected with the theory of continuants in the subject of determinants.

Sylvester shewed in 1879, "Note on Continuants," *Messenger of Mathematics*, Vol. VIII., that the number of terms in a continuant $(a_1a_2\dots a_n)$ is

$$1+\binom{n-1}{1}+\binom{n-2}{2}+\binom{n-3}{3}+\dots,$$

and that this series has the value

$$\frac{1}{\sqrt{5}}\left\{\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n+1}-\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n+1}\right\}.$$

Now recalling the result

$$V_{1,m} = 1 + x\frac{(m)}{(1)} + x^4\frac{(m-2)(m-1)}{(1)(2)} + x^9\frac{(m-4)(m-3)(m-2)}{(1)(2)(3)} + \dots,$$

we find, by putting x equal to unity, that the whole number of partitions of all numbers which have no repetitions, no sequences and no part greater than m, is

$$1+\binom{m}{1}+\binom{m-1}{2}+\binom{m-2}{3}+\dots,$$

which is equal to the number of terms in a continuant of the order $m+1$, and has therefore the value

$$\frac{1}{\sqrt{5}}\left\{\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{m+2}-\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{m+2}\right\}.$$

Similarly, from the expression for $V_{s,m}$, we find that the whole number of partitions of all numbers which have no repetitions, no sequences, and no part less than s or greater than m, is

$$1+\binom{m-s+1}{1}+\binom{m-s}{2}+\binom{m-s-1}{3}+\dots,$$

which is equal to the number of terms in a continuant of the order $m-s+2$, and has therefore the value

$$\frac{1}{\sqrt{5}}\left\{\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{m-s+3}-\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{m-s+3}\right\}.$$

In this enumeration the absolute term unity which commences the series is always included, so that the true enumeration is obtained by subtracting unity from the above expression.

When $s=m$, the only partition satisfying the conditions is m, and we find that the expression has the value 2 on development. So also when $m=2$, $s=1$, the expression has the value 3 corresponding to the *two* partitions 1, 2.

However, when $s=1$ it is, exceptionally, correct to regard the absolute partition of zero content as included, and then the expression given above is exact.

296. There is another correspondence with the enumeration of the compositions (partitions in which account is taken of the order of the parts) of numbers into parts from which unity is excluded. The function which enumerates such compositions which have exactly s parts is

$$(x^2+x^3+x^4+\dots)^s \quad \text{or} \quad \frac{x^{2s}}{(\mathbf{1})^s},$$

and the coefficient of x^w in

$$\frac{x^2}{(\mathbf{1})}+\frac{x^4}{(\mathbf{1})^2}+\frac{x^6}{(\mathbf{1})^3}+\dots$$

is

$$1+\binom{w-3}{1}+\binom{w-4}{2}+\binom{w-5}{3}+\dots,$$

which is identical with the series

$$1+\binom{m}{1}+\binom{m-1}{2}+\binom{m-2}{3}+\dots,$$

when $w=m+3$.

Hence we may say that the partitions of all numbers which have no part greater than m, and no repetitions and no sequences, are equi-numerous with the non-unitary compositions of the number $m+3$.

For $m=3$, the five-to-five correspondence is

0	1	2	3	31	partitions,
6	42	24	33	222	compositions.

There is, in fact, a more intimate correspondence still, because the number of the partitions above defined which have t parts is

$$\binom{m-t+1}{t},$$

while the number of the compositions above defined which have s parts and a content $m+3$ is

$$\binom{m-s+2}{s-1}.$$

Hence the partitions without repetitions or sequences, a highest part not greater than m and exactly s parts, are equi-numerous with the non-unitary compositions of the number $m+3$ which have exactly $s+1$ parts.

Thus for $m=7$, $s=2$ the correspondence is

31	41	42	51	52	53	61	62	63	partitions,
532	523	352	325	253	235	622	262	226	compositions,

64	71	72	73	74	75	partitions.
442	424	244	433	343	334	compositions.

(Compare Cayley, *Messenger of Mathematics*, Vol. v. (1876), pp. 166, 167.)

CHAPTER IV

PARTITIONS WITHOUT SEQUENCES

297. Just as we represented the enumerating functions of partitions without repetitions or sequences by the conditioned product

$$V_{s,m} = M(1+x^s)(1+x^{s+1})\dots(1+x^m),$$

s, m being the lower and upper limits of the part magnitude, so in the case where repetitions are allowed but not sequences we can represent the enumerating function by

$$T_{s,m} = M\frac{1}{(1-x^s)(1-x^{s+1})\dots(1-x^m)},$$

the condition of the product being that in the multiplication of any two consecutive factors

$$\frac{1}{1-x^t}\cdot\frac{1}{1-x^{t+1}},$$

viz. $$1+\frac{x^t}{1-x^t}+\frac{x^{t+1}}{1-x^{t+1}}+\frac{x^{2t+1}}{(1-x^t)(1-x^{t+1})},$$

the last term $$\frac{x^{2t+1}}{(1-x^t)(1-x^{t+1})}$$

is to be deleted.

It is best to write

$$T_{s,m} = M\left(1+\frac{x^s}{1-x^s}\right)\left(1+\frac{x^{s+1}}{1-x^{s+1}}\right)\dots\left(1+\frac{x^m}{1-x^m}\right),$$

and it will be observed that the conditioned multiplication permits repetitions but forbids sequences.

We find $$T_{s,m} = M\left(1+\frac{x^{s+1}}{1-x^{s+1}}\right)\dots\left(1+\frac{x^m}{1-x^m}\right)$$
$$+\frac{x^s}{1-x^s}M\left(1+\frac{x^{s+2}}{1-x^{s+2}}\right)\dots\left(1+\frac{x^m}{1-x^m}\right),$$

or $$T_{s,m} = T_{s+1,m} + \frac{x^s}{(s)}T_{s+2,m}.$$

Similarly $$T_{s,m} = T_{s,m-1} + \frac{x^m}{(m)} T_{s,m-2},$$

the two complementary formulæ.

In particular $$T_{1,m} = T_{2,m} + \frac{x}{(1)} T_{3,m},$$

$$T_{1,m} = T_{1,m-1} + \frac{x^m}{(m)} T_{1,m-2},$$

1 and m being now the lower and upper limits of the part magnitude.

An inclusive formula is obtained by writing

$$T_{s,m} = M\left(1 + \frac{x^s}{1-x^s}\right) \dots \left(1 + \frac{x^t}{1-x^t} + \frac{x^{t+1}}{1-x^{t+1}}\right)\left(1 + \frac{x^{t+2}}{1-x^{t+2}}\right) \dots \left(1 + \frac{x^m}{1-x^m}\right),$$

when we find

$$T_{s,m} = T_{s,t}\, T_{t+2,m} + \frac{x^{t+1}}{1-x^{t+1}} T_{s,t-1}\, T_{t+3,m}.$$

We can express $T_{s,m}$ as a linear function of $T_{1,m}$ and $T_{2,m}$.

Thus $$\frac{x}{(1)} T_{3,m} = T_{1,m} - T_{2,m},$$

$$\frac{x^3}{(1)\,(2)} T_{4,m} = -T_{1,m} + \frac{1}{(1)} T_{2,m},$$

$$\frac{x^6}{(1)\,(2)\,(3)} T_{5,m} = \frac{1}{(2)} T_{1,m} - \frac{(3)}{(1)\,(2)} T_{2,m};$$

and noticing that the last result may be written

$$\frac{x^6}{(1)\,(2)\,(3)} T_{5,m} = T_{2,2}\, T_{1,m} - T_{1,2}\, T_{2,m},$$

it is natural to assume the formula

$$\frac{x^{\binom{s-1}{2}}}{(1)\,(2) \dots (s-2)} T_{s,m} = (-)^{s+1} (T_{2,s-3}\, T_{1,m} - T_{1,s-3}\, T_{2,m}),$$

with the object of establishing it by induction.

We assume it to be true for the first s integers.

Then $$\frac{x^{\binom{s}{2}}}{(1)\,(2) \dots (s-1)} T_{s+1,m} = \frac{x^{\binom{s-1}{2}}}{(1)\,(2) \dots (s-2)} (T_{s-1,m} - T_{s,m}),$$

and substituting for $T_{s-1,m}$ and $T_{s,m}$ from the assumed formula we readily find that

$$\frac{x^{\binom{s}{2}}}{(1)\,(2) \dots (s-1)} T_{s+1,m} = (-)^s (T_{2,s-2}\, T_{1,m} - T_{1,s-2}\, T_{2,m}),$$

completing the proof.

Similarly we can obtain the complementary result; that is, we can express $T_{1,m-s}$ as a linear function of $T_{1,m}$ and $T_{1,m-1}$.

Thus in succession

$$\frac{x^m}{(m)} T_{1,m-2} = T_{1,m} - T_{1,m-1},$$

$$\frac{x^{2m-1}}{(m-1)(m)} T_{1,m-3} = -T_{1,m} + \frac{1}{(m)} T_{1,m-1},$$

$$\frac{x^{3m-3}}{(m-2)(m-1)(m)} T_{1,m-4} = \frac{1}{(m-1)} T_{1,m} - \frac{(2m-1)}{(m-1)(m)} T_{1,m-1};$$

and noticing that the last formula may be written

$$\frac{x^{3m-3}}{(m-2)(m-1)(m)} T_{1,m-4} = T_{m-1,m-1} T_{1,m} - T_{m-1,m} T_{1,m-1},$$

it is natural to assume a formula

$$\frac{x^{\frac{1}{2}(s-1)(2m-s+2)}}{(m-s+2)(m-s+3)\dots(m)} T_{1,m-s}$$
$$= (-)^s (T_{m-s+3,m-1} T_{1,m} - T_{m-s+3,m} T_{1,m-1}),$$

with the object of proving it by induction.

There is no difficulty in thus establishing it.

From the formula $\quad T_{s,m} = T_{s+1,m} + \frac{x^s}{(s)} T_{s+2,m}$

there is no difficulty in establishing the result

$$T_{s,m} = T_{s+1,m} + \frac{x^s}{(s)} T_{s+3,m} + \frac{x^{2s+2}}{(s)(s+2)} T_{s+5,m} + \frac{x^{3s+6}}{(s)(s+2)(s+4)} T_{s+7,m} + \dots,$$

and from the formula

$$T_{s,m} = T_{s,m-1} + \frac{x^m}{(m)} T_{s,m-2}$$

the result

$$T_{s,m} = T_{s,m-1} + \frac{x^m}{(m)} T_{s,m-3} + \frac{x^{2m-2}}{(m-2)(m)} T_{s,m-5}$$
$$+ \frac{x^{3m-6}}{(m-4)(m-3)(m)} T_{s,m-7} + \dots.$$

In the first series the pth term enumerates partitions which contain the parts s, $s+2$, ... $s+2p-4$, but not the part $s+2p-2$.

In the second series the pth term enumerates partitions which contain the parts m, $m-2$, ... $m-2p+4$, but not the part $m-2p+2$.

298. If we put the number of parts in the partition in evidence we introduce the symbol g and consider the coefficient of g^p in the function

$$M\left(1+\frac{gx^s}{1-gx^s}\right)\left(1+\frac{gx^{s+1}}{1-gx^{s+1}}\right)\dots\left(1+\frac{gx^m}{1-gx^m}\right).$$

Such coefficients enumerate the partitions which have exactly p parts.

Putting $s=1$ and $m=\infty$, it is easy to calculate the coefficients of the earlier powers of g. These are found to be

$$\frac{x}{(1)},$$

$$\frac{x^2}{(1)(2)}\cdot\frac{1+x^3}{1+x},$$

$$\frac{x^3}{(1)(2)(3)}\cdot\frac{1+x^3}{1+x}\cdot\frac{1+x^6}{1+x^2},$$

$$\frac{x^4}{(1)(2)(3)(4)}\cdot\frac{1+x^3}{1+x}\cdot\frac{1+x^6}{1+x^2}\cdot\frac{1+x^9}{1+x^3},$$

suggesting that the coefficient of g^p is in all probability

$$\frac{x^p}{(1)(2)\dots(p)}\cdot\frac{1+x^3}{1+x}\cdot\frac{1+x^6}{1+x^2}\cdots\frac{1+x^{3p-3}}{1+x^{p-1}},$$

an expression which may be put into the form

$$\frac{x^p}{(p)}\cdot\frac{(6)(12)(18)\dots(6p-6)}{(2)(4)(6)\dots(2p-2).(3)(6)(9)\dots(3p-3)}.$$

This shews that the function which enumerates partitions without sequences may probably be represented by the series

$$1+\frac{x}{(1)}+\frac{x^2}{(2)}\cdot\frac{(6)}{(2).(3)}+\frac{x^3}{(3)}\cdot\frac{(6)(12)}{(2)(4).(3)(6)}$$
$$+\frac{x^4}{(4)}\cdot\frac{(6)(12)(18)}{(2)(4)(6).(3)(6)(9)}+\dots.$$

In general

$$M\left(1+\frac{gx^s}{1-gx^s}\right)\left(1+\frac{gx^{s+1}}{1-gx^{s+1}}\right)\dots\left(1+\frac{gx^m}{1-gx^m}\right)\dots\text{ ad inf.}$$
$$=1+g\frac{x^s}{(1)}+g^2\frac{x^{2s}}{(2)}\cdot\frac{(6)}{(2).(3)}+g^3\frac{x^{3s}}{(3)}\cdot\frac{(6)(12)}{(2)(4).(3)(6)}+\dots.$$

299. Partitions without sequences may be looked at from another point of view. Suppose

$$(A_1A_2A_3\dots A_{p-1}A_p)$$

be a partition in which the parts are arranged in ascending order of magnitude. Denote the differences between successive parts by $\alpha_0, \alpha_1, \dots,$

$$\alpha_0A_1\alpha_1A_2\alpha_2A_3\dots A_{p-1}\alpha_{p-1}A_p.$$

If there are to be no sequences, α_0 which is equal to A_1 may be any integer but $\alpha_1, \alpha_2, \dots \alpha_{p-1}$ may each of them be zero or any integer except unity. We have

$$\begin{aligned} A_1 &= \alpha_0, \\ A_2 &= \alpha_0 + \alpha_1, \\ A_3 &= \alpha_0 + \alpha_1 + \alpha_2, \\ &\cdots\cdots\cdots\cdots \\ A_p &= \alpha_0 + \alpha_1 + \dots + \alpha_{p-1}, \end{aligned}$$

and $$\Sigma A = p\alpha_0 + (p-1)\,\alpha_1 + \dots + 2\alpha_{p-2} + \alpha_{p-1}.$$

The enumerating function is

$$\Sigma x^{p\alpha_0 + (p-1)\,\alpha_1 + \dots + 2\alpha_{p-2} + \alpha_{p-1}},$$

the summation being in regard to the values of $\alpha_0, \alpha_1, \alpha_2, \dots \alpha_{p-1}$ above specified. Now

$$\begin{aligned} \Sigma x^{\alpha_{p-1}} &= 1 + \frac{x^2}{(1)} = \frac{1+x^3}{(2)} = \frac{(6)}{(2)\,(3)}, \\ \Sigma x^{2\alpha_{p-2}} &= 1 + \frac{x^4}{(2)} = \frac{1+x^6}{(4)} = \frac{(12)}{(4)\,(6)}, \\ &\vdots \\ \Sigma x^{(p-1)\,\alpha_1} &= 1 + \frac{x^{2p-2}}{(p-1)} = \frac{1+x^{3p-3}}{(2p-2)} = \frac{(6p-6)}{(2p-2)\,(3p-3)}, \\ \Sigma x^{p\alpha_0} &= 1 + \frac{x^p}{(p)} = \frac{1}{(p)}, \end{aligned}$$

so that the function is

$$\frac{(6)}{(2)\,(3)} \cdot \frac{(12)}{(4)\,(6)} \cdots \frac{(6p-6)}{(2p-2)\,(3p-3)} \cdot \frac{1}{(p)}.$$

In this summation the smallest part may be zero. If zero be excluded as one of its values we have exactly p parts and

$$\Sigma x^{p\alpha_0} = \frac{x^p}{(p)}.$$

Hence the function which enumerates partitions without sequences and exactly p parts is

$$\frac{x^p}{(p)} \cdot \frac{(6)}{(2)\,(3)} \cdot \frac{(12)}{(4)\,(6)} \cdots \frac{(6p-6)}{(2p-2)\,(3p-3)},$$

confirming the result of the previous article.

300. If the smallest part is not to be less than two, if it be not zero,

$$\Sigma x^{p\alpha_0} = 1 + \frac{x^{2p}}{(p)} = \frac{(6p)}{(2p)\,(3p)},$$

so that the summation gives

$$\frac{(6)}{(2)\,(3)} \cdot \frac{(12)}{(4)\,(6)} \cdots \frac{(6p)}{(2p)\,(3p)},$$

which enumerates partitions of the nature under consideration which have p or fewer parts.

Putting p equal to ∞ we find now the noteworthy result that the partitions, without restriction as to the number of parts, are enumerated by the function

$$\frac{(1)(7)(13)\ldots \times (5)(11)(17)\ldots}{(1)(2)(3)\ldots}.$$

Also those which involve precisely p parts are enumerated by

$$\frac{x^{2p}}{(p)}\cdot\frac{(6)}{(2)(3)}\cdot\frac{(12)}{(4)(6)}\cdot\ldots\frac{(6p-6)}{(2p-2)(3p-3)},$$

for this is equal to the expression

$$\frac{(6)}{(2)(3)}\cdot\frac{(12)}{(4)(6)}\cdot\ldots\frac{(6p)}{(2p)(3p)}-\frac{(6)}{(2)(3)}\cdot\frac{(12)}{(4)(6)}\cdot\ldots\frac{(6p-6)}{(2p-2)(3p-3)},$$

and we reach the algebraic identity

$$1+\frac{x^2}{(1)}+\frac{x^4}{(2)}\frac{(6)}{(2)(3)}+\frac{x^6}{(3)}\frac{(6)}{(2)(3)}\cdot\frac{(12)}{(4)(6)}+\frac{x^8}{(4)}\frac{(6)}{(2)(3)}\cdot\frac{(12)}{(4)(6)}\cdot\frac{(18)}{(6)(9)}+\ldots$$

$$=\frac{(1)(7)(13)\ldots\times(5)(11)(17)\ldots}{(1)(2)(3)\ldots}$$

$$=\frac{1}{(2)(3)(4)(6).(8)(9)(10)(12).(14)(15)(16)(18)\ldots}.$$

The Partitions of Multipartite Numbers.

301. In regard to the function which enumerates the partitions of multipartite numbers in general there is not very much to be said. For the bipartite number n_1n_2 if we wish to find the number of partitions which possess i or fewer parts we seek the coefficient of $a^i x_1^{n_1} x_2^{n_2}$ in the expansion of

$$\frac{1}{(1-a)(1-ax_1)(1-ax_2)(1-ax_1^2)(1-ax_1x_2)(1-ax_2^2)\ldots},$$

wherein $1-ax_1^{p_1}x_2^{p_2}$ is a factor of the denominator, p_1, p_2 being given all integer (including zero) values.

More generally for the multipartite number $n_1n_2\ldots n_m$ the enumerating function is

$$\frac{1}{(1-a)(1-ax_1)\ldots(1-ax_m)(1-ax_1^2)(1-ax_1x_2)\ldots(1-ax_m^2)\ldots},$$

wherein $1-ax_1^{p_1}x_2^{p_2}\ldots x_m^{p_m}$ is a factor of the denominator, p_1, p_2, $\ldots$ p_m being given all integer (including zero) values.

In order to expand it in ascending powers of a we write it

$$F(a;\ x_1, x_2, \dots x_m),$$

and also

$$1 + a\phi_{m,1} + a^2\phi_{m,2} + a^3\phi_{m,3} + \dots,$$

where $\phi_{m,1}$, $\phi_{m,2}$, etc. do not involve a.

If we write ax_m for a we find at once the relation

$$F(ax_m;\ x_1, x_2, \dots x_m)\,F(a;\ x_1, x_2, \dots x_{m-1}) = F(a;\ x_1, x_2, \dots x_m),$$

leading to

$$\begin{aligned}
&(1 + ax_m\phi_{m,1} + a^2x_m^2\phi_{m,2} + a^3x_m^3\phi_{m,3} + \dots)\\
&\times(1 + a\phi_{m-1,1} + a^2\phi_{m-1,2} + a^3\phi_{m-1,3} + \dots)\\
&\quad = 1 + a\phi_{m,1} + a^2\phi_{m,2} + a^3\phi_{m,3} + \dots.
\end{aligned}$$

Comparison of coefficients of like powers of a then yields the relations

$$\begin{aligned}
&x_m\phi_{m,1} + \phi_{m-1,1} = \phi_{m,1},\\
&x_m^2\phi_{m,2} + x_m\phi_{m-1,1}\phi_{m,1} + \phi_{m-1,2} = \phi_{m,2},\\
&x_m^3\phi_{m,3} + x_m^2\phi_{m-1,1}\phi_{m,2} + x_m\phi_{m-1,2}\phi_{m,1} + \phi_{m-1,3} = \phi_{m,3},\\
&\qquad\qquad\text{etc.},
\end{aligned}$$

and we deduce the relations

$$\begin{aligned}
(1 - x_m)\,\phi_{m,1} &= \phi_{m-1,1},\\
(1 - x_m)(1 - x_m^2)\,\phi_{m,2} &= x_m\phi_{m-1,1}^2 + (1 - x_m)\,\phi_{m-1,2},\\
(1 - x_m)(1 - x_m^2)(1 - x_m^3)\,\phi_{m,3} &= x_m^3\phi_{m-1,1}^3 + (x_m + x_m^2 - 2x_m^3)\,\phi_{m-1,1}\phi_{m-1,2}\\
&\quad + (1 - x_m)(1 - x_m^2)\,\phi_{m-1,3},\\
&\text{etc.},
\end{aligned}$$

and thence

$$\phi_{m,1} = \frac{1}{(1 - x_1)(1 - x_2)\dots(1 - x_m)},$$

$$\phi_{m,2} = \frac{1 + \Sigma x_1x_2 + \Sigma x_1x_2x_3x_4 + \Sigma x_1x_2x_3x_4x_5x_6 + \dots}{(1 - x_1)(1 - x_1^2).(1 - x_2)(1 - x_2^2)\dots.(1 - x_m)(1 - x_m^2)},$$

where in the last fraction the elementary symmetric functions of even order of the magnitudes $x_1, x_2, \dots x_m$ appear.

The functions $\phi_{m,3}$, etc. are very complex and lead to no interesting results.

The result in the case of $\phi_{m,1}$ is of a trivial character.

302. The case of $\phi_{m,2}$ is however worth a moment's consideration.

It enumerates the partitions of all multipartite numbers into two or fewer parts. Its form is such that the enumeration is made to depend upon the enumeration of the partitions of unipartite numbers into two or fewer parts.

Let
$$B_{n_1 n_2 \dots n_m}$$
denote the number of partitions, into two or fewer parts, of the multipartite number $n_1 n_2 \dots n_m$. Then the form of $\phi_{m,2}$ shews that

$$\begin{aligned} B_{n_1 n_2 \dots n_m} = B_{n_1} B_{n_2} \dots B_{n_m} \\ + \Sigma B_{n_1 - 1} B_{n_2 - 1} B_{n_3} \dots B_{n_m} \\ + \Sigma B_{n_1 - 1} B_{n_2 - 1} B_{n_3 - 1} B_{n_4 - 1} B_{n_5} \dots B_{n_m} \\ + \dots\dots, \end{aligned}$$

the symmetric function in the sth line involving $\binom{m}{2s-2}$ terms.

We shall verify this result for the case

$$n_1 = n_2 = \dots = n_m = n.$$

This particular case is convenient because we can predict the result with ease. If a partition into two parts be

$$(\nu_1 \nu_2 \dots \nu_m)(\nu_1' \nu_2' \dots \nu'_m),$$

the constituent ν_s may be any one of the numbers

$$0,\ 1,\ 2,\ \dots n$$

and
$$\nu_s + \nu_s' = n.$$

The case in which $\nu_1 = \nu_2 = \dots = \nu_m = 0$ is included, because then in fact we have a partition into one part included in the enumeration. Hence ν_s may have $n+1$ different values and the value of ν_s determines the value of ν_s'. It follows that

$$(\nu_1 \nu_2 \dots \nu_m)$$

may have $(n+1)^m$ different values and this fact leads to

$$\tfrac{1}{2}(n+1)^m,$$

or
$$\tfrac{1}{2}(n+1)^m + \tfrac{1}{2}$$

partitions into two parts according as n is uneven or even.

The reader will notice that when n is uneven every partition occurs twice in the above process, but that when n is even one of the partitions occurs only once. This happens when

$$\nu_s = \nu_s' = \tfrac{1}{2}n$$

for all values of s.

We now observe that

$$B_s = \tfrac{1}{2}(s+1) \text{ or } \tfrac{1}{2}(s+2),$$

according as s is uneven or even.

The formula therefore gives

CASE I. n uneven:

$$\{\tfrac{1}{2}(n+1)\}^m + \binom{m}{2}\{\tfrac{1}{2}(n+1)\}^{m-2}\{\tfrac{1}{2}(n+1)\}^2 + \binom{m}{4}\{\tfrac{1}{2}(n+1)\}^{m-4}\{\tfrac{1}{2}(n+1)\}^4 + \dots$$

$$= \frac{1}{2^m}(n+1)^m\left\{1 + \binom{m}{2} + \binom{m}{4} + \dots\right\}$$

$$= \tfrac{1}{2}(n+1)^m,$$

which is correct.

CASE II. n even: the number is

$$\{\tfrac{1}{2}(n+2)\}^m + \binom{m}{2}\{\tfrac{1}{2}(n+2)\}^{m-2}\{\tfrac{1}{2}n\}^2 + \binom{m}{4}\{\tfrac{1}{2}(n+2)\}^{m-4}\{\tfrac{1}{2}n\}^4 + \dots.$$

Herein we write

$$\binom{m}{2s}\{\tfrac{1}{2}(n+2)\}^{m-2s}\{\tfrac{1}{2}n\}^{2s}$$

$$= \binom{m}{2s}\{\tfrac{1}{2}(n+2)^{m-2s}\}\{\tfrac{1}{2}(n+2)-1\}^{2s},$$

and it is now an easy task to shew that the expression is equal to

$$\tfrac{1}{2}\left\{(n+2)^m - \binom{m}{1}(n+2)^{m-1} + \binom{m}{2}(n+2)^{m-2} - \dots (-)^m 1 + 1\right\},$$

or to $\qquad \tfrac{1}{2}\{(n+2-1)^m + 1\}$, or to $\tfrac{1}{2}(n+1)^m + \tfrac{1}{2}$.

For consider the identity

$$\{p + x(p-1)\}^m = p^m + \binom{m}{1}p^{m-1}x(p-1) + \binom{m}{2}p^{m-2}x^2(p-1)^2 + \dots;$$

put successively x equal to $+1$ and to -1 and add, obtaining

$$(2p-1)^m + 1 = 2\left\{p^m + \binom{m}{2}p^{m-2}(p-1)^2 + \binom{m}{4}p^{m-4}(p-1)^4 + \dots\right\},$$

and now putting $p = \tfrac{1}{2}(n+2)$ we have proved as required.

303. The connexion between the partitions and divisors of numbers was established by Euler*.

He proceeds from his formula

$$(1-x)(1-x^2)(1-x^3)\dots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots.$$

Differentiating logarithmically and multiplying throughout by $-x$ we find

$$\frac{x}{1-x} + 2\frac{x^2}{1-x^2} + 3\frac{x^3}{1-x^3} + \dots = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \dots}{(1-x)(1-x^2)(1-x^3)\dots}.$$

* "Observatio de summis divisorum," *Opera Minora Collecta*, Vol. I. pp. 141—154.

On the left-hand side the coefficient of x^n is equal to the sum of the divisors of n. Denoting it by $\sigma(n)$ and the number of partitions of n by $P(n)$ we have

$$\begin{aligned}&\sigma(1)\,x+\sigma(2)\,x^2+\sigma(3)\,x^3+\dots\\&=(x+2x^2-5x^5-7x^7+12x^{12}+15x^{15}+\dots)\\&\times(1+P(1)\,x+P(2)\,x^2+P(3)\,x^3+\dots),\end{aligned}$$

and by equating coefficients of like powers of x

$$\begin{aligned}\sigma(n)=P(n-1)+2P(n-2)-5P(n-5)-7P(n-7)\\+12P(n-12)+15P(n-15)+\dots,\end{aligned}$$

the two general terms on the right-hand side being

$$(-)^{r+1}\,\tfrac{1}{2}r\,(3r-1)\,P\,\{n-\tfrac{1}{2}r\,(3r-1)\},$$
$$(-)^{r+1}\,\tfrac{1}{2}r\,(3r+1)\,P\,\{n-\tfrac{1}{2}r\,(3r+1)\}.$$

Writing the intermediate formula in the form

$$\begin{aligned}\{\sigma(1)\,x+\sigma(2)\,x^2+\sigma(3)\,x^3+\dots\}\,(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\dots)\\=x+2x^2-5x^5-7x^7+12x^{12}+15x^{15}+\dots\end{aligned}$$

we obtain another of Euler's results

$$\sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\dots=0\quad\text{or}\quad(-)^{r+1}\,n,$$

according as n is not or is a pentagonal number

$$\tfrac{1}{2}r\,(3r\pm 1).$$

CHAPTER V

PARTICULAR STUDY OF THE FUNCTION

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})}{(1-x)(1-x^2)\dots(1-x^i)}$$

304. The study upon which we now enter has for its object the transformation of the enumerating generating function of partitions limited both as to number of parts and as to part magnitude, so that it may be conveniently expanded.

We first have to note the method of Cayley, whose purpose was to transform it so that it might be dealt with by breaking it up into partial fractions.

Cayley's Transformation.*

We take the equation

$$(1+ax)(1+ax^2)\dots(1+ax^i)$$
$$=1+ax\frac{(1-x^i)}{1-x}+a^2x^3\frac{(1-x^{i-1})(1-x^i)}{(1-x)(1-x^2)}+\dots,$$

where the general term is

$$a^s x^{\frac{1}{2}s(s+1)}\frac{(1-x^{i-s+1})(1-x^{i-s+2})\dots(1-x^i)}{(1-x)(1-x^2)\dots(1-x^s)}.$$

The series is finite, having $i+1$ terms, the last term being

$$a^i x^{\frac{1}{2}i(i+1)}.$$

We now write $-x^j$ for a and obtain the identity

$$(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})$$
$$=1-x^{j+1}\frac{(1-x^i)}{(1-x)}+x^{2j+3}\frac{(1-x^{i-1})(1-x^i)}{(1-x)(1-x^2)}+\dots$$
$$=\sum_{s=0}^{s=i}(-)^s x^{sj+\frac{1}{2}s(s+1)}\frac{(1-x^{i-s+1})(1-x^{i-s+2})\dots(1-x^i)}{(1-x)(1-x^2)\dots(1-x^s)}.$$

* *Phil. Trans. R. S. London*, Vol. CXLV. 1855.

Dividing both sides by $(1-x)(1-x^2)\dots(1-x^i)$ we find

$$\frac{(1-x^{j+1})(1-x^{j+2})\dots(1-x^{j+i})}{(1-x)(1-x^2)\dots(1-x^i)}$$

$$=\sum_{s=0}^{s=i}(-)^s x^{sj+\frac{1}{2}s(s+1)}\frac{1}{(1-x)(1-x^2)\dots(1-x^s).(1-x)(1-x^2)\dots(1-x^{i-s})}.$$

The function under examination is thus presented in the form of a sum of $i+1$ fractions each of which can be dealt with by the method of partial fractions.

305. For full particulars of the way in which Cayley employs partial fractions the reader may consult the paper quoted. It is convenient here to explain the notation in which he exhibits his results.

The function $$\frac{1}{(1-x)(1-x^2)}$$

is in the ordinary manner thrown into the form

$$\frac{1}{2}\frac{1}{(1-x)^2}+\frac{1}{4}\frac{1}{1-x}+\frac{1}{4}\frac{1}{1+x},$$

and supposing that the coefficient of x^w in the function is required, he denotes this coefficient by

$$P(1, 2)\,w.$$

He then expands each of the three fractions and finds that the coefficients of x^w in the first and second fractions are

$$\tfrac{1}{2}(w+1) \text{ and } \tfrac{1}{4} \text{ respectively.}$$

To expand the third fraction he writes it

$$\frac{1}{4}\frac{1-x}{1-x^2},$$

and denotes the coefficient of x^w by the notation

$$\tfrac{1}{4}(1,-1)\text{ pcr } 2_w,$$

wherein the portion $(1,-1)$ has reference to the numerator coefficient, pcr means "prime circulator," and the combined expression

$$(1,-1)\text{ pcr } 2_w$$

indicates the way in which the coefficient of x^w is actually obtained by expansion.

He thus gives the formula

$$P(1, 2)\,w=\frac{1}{4}\left\{\begin{array}{c}2w+3\\+(1,-1)\text{ pcr } 2_w\end{array}\right\},$$

and in a precisely similar manner he obtains the formula derived from

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

This is

$$P(1, 2, 3)\, w = \frac{1}{72}\begin{Bmatrix} 6w^2 + 36w + 47 \\ + 9(1, -1) \text{ pcr } 2_w \\ + 8(2, -1, -1) \text{ pcr } 3_w \end{Bmatrix},$$

wherein $(2, -1, -1) \text{ pcr } 3_w$

is derived from $$\frac{2+x}{1+x+x^2} = \frac{2-x-x^2}{1-x^3}.$$

306. The notation is convenient, but a very slight consideration will convince the reader that the method does not always exhibit results in the simplest form.

For example, Cayley applies the method to the simple fraction

$$\frac{1}{1-x^2},$$

and finds $P(2)\, w$ (viz. the number of ways of partitioning w into parts when 2 is the only part allowed) to have the expression

$$\frac{1}{2}\begin{Bmatrix} 1 \\ + (1, -1) \text{ pcr } 2_w \end{Bmatrix}.$$

Now we know that with the notation explained

$$P(2)\, w = (1, 0) \text{ pcr } 2_w,$$

so that clearly the ordinary method of partial fractions cannot be relied upon to give the result invariably in the simplest (and therefore the best) form.

Transformation by Symmetric Functions.

307. We commence by observing the two identities

$$\frac{1}{(1-x)(1-x^2)} = \frac{1}{2}\frac{1}{(1-x)^2} + \frac{1}{2}\frac{1}{(1-x^2)},$$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{6}\frac{1}{(1-x)^3} + \frac{1}{2}\frac{1}{(1-x)(1-x^2)} + \frac{1}{3}\frac{1}{1-x^3},$$

which we will also write in the illuminating notation so often employed:

$$\frac{1}{\mathbf{(1)\,(2)}} = \frac{1}{2}\frac{1}{\mathbf{(1)}^2} + \frac{1}{2}\frac{1}{\mathbf{(2)}},$$

$$\frac{1}{\mathbf{(1)\,(2)\,(3)}} = \frac{1}{6}\frac{1}{\mathbf{(1)}^3} + \frac{1}{2}\frac{1}{\mathbf{(2)\,(1)}} + \frac{1}{3}\frac{1}{\mathbf{(3)}}.$$

The observation leads to the conjecture that we are in the presence of partial fractions of a new and special kind. We note that in the first identity we have a fraction corresponding to each of the partitions $(1)^2$, (2) of the

number 2 and in the second fractions corresponding to and derived from each of the partitions $(1)^3$, (21), (3) of the number 3. The coefficients of the fractions also remind us of numbers which arise in symmetric function formulæ, and we are led easily to the symmetric function view of the matter which is now given.

Consider symmetric functions of the powers of x, infinite in number, viz.

$$1,\ x,\ x^2,\ x^3,\ \ldots.$$

Let s_i denote the sum of the ith powers of these quantities. Then clearly

$$s_i = \frac{1}{1-x^i} = \frac{1}{(\mathrm{i})}.$$

Now if h_i be the homogeneous product sum of weight i of the quantities $\alpha, \beta, \gamma, \ldots$, we have

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)\ldots} = 1 + h_1 z + h_2 z^2 + \ldots + h_i z^i + \ldots.$$

Putting herein $1, x, x^2, \ldots$ instead of $\alpha, \beta, \gamma, \ldots$ we have

$$\frac{1}{(1-z)(1-xz)(1-x^2z)\ldots} = 1 + \frac{z}{(1)} + \frac{z^2}{(1)\,(2)} + \ldots + \frac{z^i}{(1)\,(2)\ldots(\mathrm{i})} + \ldots$$

Hence if h_i denote the homogeneous product sum of order i of the quantities

$$1,\ x,\ x^2,\ x^3,\ \ldots$$

we get
$$h_i = \frac{1}{(1)\,(2)\,\ldots\,(\mathrm{i})}.$$

Having obtained expressions for s_i and h_i we recall that we can express h_i in terms of $s_1, s_2, s_3, \ldots$ (see Vol. I, Section I, Chapter I, Art. 6). The formula is

(i)
$$h_i = \Sigma\, \frac{s_1^{p_1} s_2^{p_2} s_3^{p_3}\ldots}{1^{p_1}.\,2^{p_2}.\,3^{p_3}\ldots p_1!\,p_2!\,p_3!\ldots},$$

where $(1^{p_1}2^{p_2}3^{p_3}\ldots)$ is a partition of i and the summation is in regard to all partitions of i.

Substituting herein

$$\frac{1}{(1)\,(2)\ldots(\mathrm{i})},\quad \frac{1}{(1)},\quad \frac{1}{(2)},\quad \frac{1}{(3)},\ \ldots$$

for
$$h_i,\qquad s_1,\quad s_2,\quad s_3,\ \ldots$$

respectively we find

$$\frac{1}{(1)\,(2)\ldots(\mathrm{i})} = \Sigma\, \frac{1}{1^{p_1}.\,2^{p_2}.\,3^{p_3}\ldots p_1!\,p_2!\,p_3!\ldots}\;\frac{1}{(1)^{p_1}\,(2)^{p_2}\,(3)^{p_3}\ldots},$$

where $(1^{p_1}2^{p_2}3^{p_3}\ldots)$ is a partition of i and the summation is in regard to all partitions of i.

308. This remarkable result shews the decomposition of the generating function into as many fractions as the number i possesses partitions. The denominator of each fraction is directly derived from one of the partitions and is of degree i in x. The numerator does not involve x and the coefficient is the easily calculable number

$$\frac{1}{1^{p_1} \cdot 2^{p_2} \cdot 3^{p_3} \dots p_1! \, p_2! \, p_3! \dots}.$$

The general formula, on proceeding to the coefficient of x^w, takes the form in Cayley's Notation

$$P(1, 2, 3, \dots i)\, w = \Sigma \frac{P(1^{p_1} 2^{p_2} 3^{p_3} \dots)\, w}{1^{p_1} \cdot 2^{p_2} \cdot 3^{p_3} \dots p_1! \, p_2! \, p_3! \dots}.$$

We have used the two symmetric functions s_i, h_i.

For $i = 2$ we get

$$\frac{1}{(1)\,(2)} = \frac{1}{2} \frac{1}{(1)^2} + \frac{1}{2} \frac{1}{(2)}$$

and we are led, in Cayley's notation, to the result

$$P(1, 2)\, w = \frac{1}{2} \begin{Bmatrix} w + 1 \\ + (1, 0) \text{ per } 2_w \end{Bmatrix}.$$

Also for $i = 3$ we have

$$\frac{1}{(1)\,(2)\,(3)} = \frac{1}{6} \frac{1}{(1)^3} + \frac{1}{2} \frac{1}{(2)\,(1)} + \frac{1}{3} \frac{1}{(3)},$$

or making use of the decomposition of $\dfrac{1}{(1)\,(2)}$,

$$\frac{1}{(1)\,(2)\,(3)} = \frac{1}{6} \frac{1}{(1)^3} + \frac{1}{4} \frac{1}{(1)^2} + \frac{1}{4} \frac{1}{(2)} + \frac{1}{3} \frac{1}{(3)},$$

leading to

$$P(1, 2, 3)\, w = \frac{1}{12} \begin{Bmatrix} (w + 1)(w + 5) \\ + 3 (1, 0) \text{ per } 2_w \\ + 4 (1, 0, 0) \text{ per } 3_w \end{Bmatrix},$$

expressions which are simpler than those given by Cayley.

Of the order 4 similarly

$$\frac{1}{(1)\,(2)\,(3)\,(4)} = \frac{1}{24} \frac{1}{(1)^4} + \frac{1}{4} \frac{1}{(2)\,(1)^2} + \frac{1}{8} \frac{1}{(2)^2} + \frac{1}{3} \frac{1}{(3)\,(1)} + \frac{1}{4} \frac{1}{(4)}.$$

As is well known the sum of the coefficients on the right-hand side is unity. This serves as a verification.

Of these fractions $\dfrac{1}{(1)^4}$, $\dfrac{1}{(2)^2}$, $\dfrac{1}{(4)}$ are immediately dealt with.

Moreover,
$$\frac{1}{(2)(1)^2}=\frac{1}{(1)}\left\{\frac{1}{2}\frac{1}{(1)^2}+\frac{1}{2}\frac{1}{(2)}\right\}$$
$$=\frac{1}{2}\frac{1}{(1)^3}+\frac{1}{4}\frac{1}{(1)^2}+\frac{1}{4}\frac{1}{(2)},$$
$$\frac{1}{(3)(1)}=\frac{1+x+x^2}{(3)^2}=(1,1,1)\text{ pcr }3^2_w,$$
and the expression of $P(1, 2, 3, 4)w$ is readily written down.

It thus appears that the Transformation by Symmetric Functions is a valuable first step towards the expansion of the function. Particular fractions that thence arise may require breaking up by the same or some other process as a second step.

309. The fraction
$$\frac{1}{(1-x)(1-x^2)\dots(1-x^s).(1-x)(1-x^2)\dots(1-x^{i-s})},$$
which presents itself in Cayley's Transformation, is seen to be
$$h_s h_{i-s},$$
and is therefore expressible in terms of
$$\frac{1}{(1)},\ \frac{1}{(2)},\ \frac{1}{(3)},\dots$$
by the above formula. Each fraction thus obtained will have a denominator of weight i.

310. The function a_i, that is to say, the sum of the quantities i and i together of the quantities
$$1,\quad x,\quad x^2,\dots,$$
is obtained from the formula
$$(1+x^0)(1+x)(1+x^2)(1+x^3)\dots=1+\frac{1}{(1)}+\frac{x}{(1)(2)}+\frac{x^3}{(1)(2)(3)}+\dots,$$
yielding
$$a_i=\frac{x^{\frac{1}{2}i(i-1)}}{(1)(2)\dots(i)},$$
and now the three functions give rise to six formulæ, of which one has already been given. The five others are:

(ii)
$$s_i=\Sigma(-)^{\Sigma p-1}\frac{(\Sigma p-1)!\,i}{p_1!\,p_2!\,p_3!\dots}h_1^{p_1}h_2^{p_2}h_3^{p_3}\dots,$$
which, if we denote
$$\frac{1}{(1)(2)\dots(i)}\text{ for brevity by }\frac{1}{[i]},$$
leads to
$$\frac{1}{(i)}=\Sigma(-)^{\Sigma p-1}\frac{(\Sigma p-1)!\,i}{p_1!\,p_2!\,p_3!\dots}\frac{1}{[1]^{p_1}}\frac{1}{[2]^{p_2}}\frac{1}{[3]^{p_3}}\dots,$$

where $(1^{p_1} 2^{p_2} 3^{p_3} \dots)$ is a partition of i, and the summation is for all partitions of i;

(iii) $$a_i = \Sigma (-)^{i+\Sigma p} \frac{s_1^{p_1} s_2^{p_2} s_3^{p_3} \dots}{1^{p_1} . 2^{p_2} . 3^{p_3} \dots p_1! p_2! p_3! \dots},$$

leading to

$$\frac{x^{\frac{1}{2}i(i-1)}}{(1)(2)\dots(i)} = \Sigma \frac{(-)^{i+\Sigma p}}{1^{p_1} . 2^{p_2} . 3^{p_3} \dots p_1! p_2! p_3! \dots} \cdot \frac{1}{(1)^{p_1} (2)^{p_2} (3)^{p_3} \dots};$$

(iv) $$s_i = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p - 1)!\, i}{p_1! p_2! p_3! \dots} a_1^{p_1} a_2^{p_2} a_3^{p_3} \dots,$$

leading to

$$\frac{1}{(i)} = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p - 1)!\, i}{p_1! p_2! p_3! \dots} \left\{\frac{1}{[1]}\right\}^{p_1} \left\{\frac{x}{[2]}\right\}^{p_2} \left\{\frac{x^3}{[3]}\right\}^{p_3} \dots;$$

(v) $$h_i = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p)!}{p_1! p_2! p_3! \dots} a_1^{p_1} a_2^{p_2} a_3^{p_3} \dots,$$

leading to

$$\frac{1}{[i]} = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p)!}{p_1! p_2! p_3! \dots} \left\{\frac{1}{[1]}\right\}^{p_1} \left\{\frac{x}{[2]}\right\}^{p_2} \left\{\frac{x^3}{[3]}\right\}^{p_3} \dots;$$

(vi) $$a_i = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p)!}{p_1! p_2! p_3! \dots} h_1^{p_1} h_2^{p_2} h_3^{p_3} \dots,$$

leading to

$$\frac{x^{\frac{1}{2}i(i-1)}}{[i]} = \Sigma (-)^{i+\Sigma p} \frac{(\Sigma p)!}{p_1! p_2! p_3! \dots} \frac{1}{[1]^{p_1} [2]^{p_2} [3]^{p_3} \dots}.$$

These all involve noteworthy theorems in partitions.

The formula in (iii) above may be expressed in the form

$$P(1, 2, 3, \dots i)\{w - \tfrac{1}{2}i(i-1)\} = \Sigma (-)^{i+\Sigma p} \frac{P(1^{p_1} 2^{p_2} 3^{p_3} \dots)\, w}{1^{p_1} . 2^{p_2} . 3^{p_3} \dots p_1! p_2! p_3! \dots}.$$

311. We have next to extend these results by considering the symmetric functions of the finite number of powers of x,

$$1, \quad x, \quad x^2, \dots x^j.$$

We find $$s_i = \frac{1 - x^{ij+i}}{1 - x^i} = \frac{(ij+i)}{(i)}.$$

The formula $$\frac{1}{(1-z)(1-xz)(1-x^2z)\dots(1-x^jz)}$$

$$= 1 + z\frac{(j+1)}{(1)} + z^2 \frac{(j+1)(j+2)}{(1)(2)} + \dots$$

shews that $$h_i = \frac{(j+1)(j+2)\dots(j+i)}{(1)(2)\dots(i)},$$

and the formula

$$(1+z)(1+xz)(1+x^2z)\dots(1+x^jz)$$

$$=1+z\frac{(\mathrm{j}+1)}{(1)}+z^2x\frac{(\mathrm{j})\,(\mathrm{j}+1)}{(1)\,(2)}+z^3x^3\frac{(\mathrm{j}-1)\,(\mathrm{j})\,(\mathrm{j}+1)}{(1)\,(2)\,(3)}+\dots,$$

that
$$a_i=x^{\frac{1}{2}i(i-1)}\frac{(\mathrm{j}-\mathrm{i}+2)\,(\mathrm{j}-\mathrm{i}+3)\dots(\mathrm{j}+1)}{(1)\,(2)\dots(\mathrm{i})}.$$

We are accordingly led to the formulæ

312. (i)
$$\frac{(\mathrm{j}+1)\,(\mathrm{j}+2)\dots(\mathrm{j}+\mathrm{i})}{(1)\,(2)\dots(\mathrm{i})}$$
$$=\Sigma\frac{1}{1^{p_1}.2^{p_2}.3^{p_3}\dots p_1!\,p_2!\,p_3!\dots}\left\{\frac{(\mathrm{j}+1)}{(1)}\right\}^{p_1}\left\{\frac{(2\mathrm{j}+2)}{(2)}\right\}^{p_2}\left\{\frac{(3\mathrm{j}+3)}{(3)}\right\}^{p_3}\dots.$$

Here the left-hand side enumerates partitions which are limited to involve not more than j parts, and no part greater than i.

The product
$$\left\{\frac{(\mathrm{j}+1)}{(1)}\right\}^{p_1}\left\{\frac{(2\mathrm{j}+2)}{(2)}\right\}^{p_2}\left\{\frac{(3\mathrm{j}+3)}{(3)}\right\}^{p_3}\dots,$$

which is typical of those on the right-hand side, enumerates partitions whose parts may be

ones	of p_1	different kinds
twos	„ p_2	„
threes	„ p_3	„

....

but in which no part of a particular kind occurs more than j times.

The simplest cases of the formula are

$$\frac{(\mathrm{j}+1)\,(\mathrm{j}+2)}{(1)\,(2)}=\frac{1}{2}\frac{(\mathrm{j}+1)^2}{(1)^2}+\frac{1}{2}\frac{(2\mathrm{j}+2)}{(2)},$$

$$\frac{(\mathrm{j}+1)\,(\mathrm{j}+2)\,(\mathrm{j}+3)}{(1)\,(2)\,(3)}=\frac{1}{6}\frac{(\mathrm{j}+1)^3}{(1)^3}+\frac{1}{2}\frac{(2\mathrm{j}+2)\,(\mathrm{j}+1)}{(2)\,(1)}+\frac{1}{3}\frac{(3\mathrm{j}+3)}{(3)},$$

and if, extending Cayley's notation, we denote the coefficient of x^w in

$$\frac{(\mathrm{j}+1)\,(\mathrm{j}+2)\dots(\mathrm{j}+\mathrm{i})}{(1)\,(2)\dots(\mathrm{i})}$$

by
$$P(1, 2, 3, \dots i;\ j)\,w,$$

and the coefficient of x^w in

$$\left\{\frac{(\mathrm{j}+1)}{(1)}\right\}^{p_1}\left\{\frac{(2\mathrm{j}+2)}{(2)}\right\}^{p_2}\left\{\frac{(3\mathrm{j}+3)}{(3)}\right\}^{p_3}\dots$$

by
$$P_j(1^{p_1}, 2^{p_2}, 3^{p_3}, \dots)\,w,$$

we have the relation

$$P(1, 2, 3, \dots i;\ j)\,w=\Sigma\frac{P_j(1^{p_1}, 2^{p_2}, 3^{p_3}, \dots)\,w}{1^{p_1}.2^{p_2}.3^{p_3}\dots p_1!\,p_2!\,p_3!\dots},$$

in itself a noteworthy theorem in partitions.

If in the formula we put $x=1$, we obtain the arithmetical relation

$$\binom{j+i}{i}=\Sigma\,\frac{(j+1)^{p_1+p_2+p_3+\dots}}{1^{p_1}.\,2^{p_2}.\,3^{p_3}\dots p_1!\,p_2!\,p_3!\dots},$$

a theorem which expresses the binomial coefficient

$$\binom{j+i}{i}$$

as a linear function of powers of the integer $j+1$.

The simplest cases are

$$\binom{j+2}{2}=\tfrac{1}{2}(j+1)^2+\tfrac{1}{2}(j+1),$$

$$\binom{j+3}{3}=\tfrac{1}{6}(j+1)^3+\tfrac{1}{2}(j+1)^2+\tfrac{1}{3}(j+1).$$

313. (ii) Writing for brevity

$$\frac{(\mathbf{j+1})\,(\mathbf{j+2})\dots(\mathbf{j+i})}{(\mathbf{1})\,(\mathbf{2})\dots(\mathbf{i})}=\left[\frac{\mathrm{j+i}}{\mathrm{i}}\right],$$

we obtain

$$\frac{(\mathbf{ij+i})}{(\mathbf{i})}=\Sigma\,(-)^{\Sigma p-1}\,\frac{(\Sigma p-1)!\,i}{p_1!\,p_2!\,p_3!\dots}\left[\frac{\mathrm{j+1}}{1}\right]^{p_1}\left[\frac{\mathrm{j+2}}{2}\right]^{p_2}\left[\frac{\mathrm{j+3}}{3}\right]^{p_3}\dots.$$

The product

$$\left[\frac{\mathrm{j+1}}{1}\right]^{p_1}\left[\frac{\mathrm{j+2}}{2}\right]^{p_2}\left[\frac{\mathrm{j+3}}{3}\right]^{p_3}\dots$$

is

$$\left\{\frac{(\mathbf{j+1})}{(\mathbf{1})}\right\}^{p_1+p_2+p_3+\dots}.\left\{\frac{(\mathbf{j+2})}{(\mathbf{2})}\right\}^{p_2+p_3+\dots}.\left\{\frac{(\mathbf{j+3})}{(\mathbf{3})}\right\}^{p_3+\dots}\dots.$$

If we put x equal to unity we have the arithmetical relation

$$j+1=\Sigma\,(-)^{\Sigma p-1}\,\frac{(\Sigma p-1)!\,i}{p_1!\,p_2!\,p_3!\dots}\binom{j+1}{1}^{p_1}\binom{j+2}{2}^{p_2}\binom{j+3}{3}^{p_3}\dots,$$

which expresses the integer $j+1$ as a linear function of certain definite products of binomial coefficients.

The simplest cases are

$$j+1=-\binom{j+1}{1}^2+2\binom{j+2}{2},$$

$$j+1=\binom{j+1}{1}^3-3\binom{j+2}{2}\binom{j+1}{1}+3\binom{j+3}{3}.$$

314. (iii)
$$x^{\frac{1}{2}i(i-1)}\,\frac{(\mathbf{j-i+2})\,(\mathbf{j-i+3})\dots(\mathbf{j+1})}{(\mathbf{1})\,(\mathbf{2})\dots(\mathbf{i})}$$

$$=\Sigma\,(-)^{i+\Sigma p}\,\frac{\left\{\frac{(\mathbf{j+1})}{(\mathbf{1})}\right\}^{p_1}\left\{\frac{(\mathbf{2j+2})}{(\mathbf{2})}\right\}^{p_2}\left\{\frac{(\mathbf{3j+3})}{(\mathbf{3})}\right\}^{p_3}\dots}{1^{p_1}.\,2^{p_2}.\,3^{p_3}\dots p_1!\,p_2!\,p_3!\dots}.$$

The factor

$$\frac{(\mathbf{j}-\mathbf{i}+2)(\mathbf{j}-\mathbf{i}+3)\dots(\mathbf{j}+1)}{(1)(2)\dots(\mathbf{i})}$$

may also be written in the form

$$\frac{(1)(2)\dots(\mathbf{j}+1)}{(1)(2)\dots(\mathbf{i}).(1)(2)\dots(\mathbf{j}-\mathbf{i}+1)}.$$

It is obvious that, since a_i refers to the quantities

$$1,\ x,\ x^2, \dots x^j,$$

i must be equal to or less than $j+1$.

The factor enumerates partitions into parts limited in number to $j-i+1$ and in magnitude to i *or* partitions limited in number to i and in magnitude to $j-i+1$.

We may therefore write

$$P(1, 2, 3, \dots i;\ j-i+1)\{w-\tfrac{1}{2}i(i-1)\}$$
$$=\Sigma(-)^{i+\Sigma p}\frac{P_j(1^{p_1}, 2^{p_2}, 3^{p_3}, \dots)w}{1^{p_1}.2^{p_2}.3^{p_3}.\dots p_1!p_2!p_3!\dots},$$

involving a noteworthy theorem.

The simplest cases are

$$x\frac{(\mathbf{j})(\mathbf{j}+1)}{(1)(2)}=\frac{1}{2}\frac{(\mathbf{j}+1)^2}{(1)^2}-\frac{1}{2}\frac{(2\mathbf{j}+2)}{(2)},$$

$$x^3\frac{(\mathbf{j}-1)(\mathbf{j})(\mathbf{j}+1)}{(1)(2)(3)}=\frac{1}{6}\frac{(\mathbf{j}+1)^3}{(1)^3}-\frac{1}{2}\frac{(2\mathbf{j}+2)}{(2)}\frac{(\mathbf{j}+1)}{(1)}+\frac{1}{3}\frac{(3\mathbf{j}+3)}{(3)}.$$

Putting x equal to unity we have the arithmetical relation

$$\binom{j+1}{i}=\Sigma(-)^{i+\Sigma p}\frac{(j+1)^{\Sigma p}}{1^{p_1}.2^{p_2}.3^{p_3}\dots p_1!p_2!p_3!\dots},$$

of which the simplest cases are

$$\binom{j+1}{2}=\tfrac{1}{2}(j+1)^2-\tfrac{1}{2}(j+1),$$

$$\binom{j+1}{3}=\tfrac{1}{6}(j+1)^3-\tfrac{1}{2}(j+1)^2+\tfrac{1}{3}(j+1).$$

315. (iv) We have

$$\frac{(\mathbf{ij}+\mathbf{i})}{(\mathbf{i})}=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p-1)!\,i}{p_1!p_2!p_3!\dots}x^{p_2+3p_3+\dots}\left[\frac{\mathbf{j}+1}{1}\right]^{p_1}\left[\frac{\mathbf{j}+1}{2}\right]^{p_2}\left[\frac{\mathbf{j}+1}{3}\right]^{p_3}\dots.$$

The product $$\left[\frac{\mathbf{j}+1}{1}\right]^{p_1}\left[\frac{\mathbf{j}+1}{2}\right]^{p_2}\left[\frac{\mathbf{j}+1}{3}\right]^{p_3}\dots$$

is $$\left\{\frac{(\mathbf{j}+1)}{(1)}\right\}^{p_1+p_2+p_3+\dots}\left\{\frac{(\mathbf{j})}{(2)}\right\}^{p_2+p_3+\dots}\left\{\frac{(\mathbf{j}-1)}{(3)}\right\}^{p_3+\dots}\dots.$$

Putting x equal to unity we have the arithmetical theorem

$$j+1=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p-1)!\,i}{p_1!\,p_2!\,p_3!\ldots}\binom{j+1}{1}^{p_1}\binom{j+1}{2}^{p_2}\binom{j+1}{3}^{p_3}\ldots,$$

with the simplest cases

$$j+1=\binom{j+1}{1}^2-2\binom{j+1}{2},$$

$$j+1=\binom{j+1}{1}^3-3\binom{j+1}{2}\binom{j+1}{1}+3\binom{j+1}{3}.$$

316. (v) $$\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{i}\end{matrix}\right]=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p)!}{p_1!\,p_2!\,p_3!\ldots}\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{1}\end{matrix}\right]^{p_1}\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{2}\end{matrix}\right]^{p_2}\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{3}\end{matrix}\right]^{p_3}\ldots$$
$$\times\, x^{p_2+3p_3+\ldots+\frac{1}{2}i(i-1)p_i+\ldots}.$$

Putting x equal to unity we have the arithmetical result

$$\binom{j+i}{i}=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p)!}{p_1!\,p_2!\,p_3!\ldots}\binom{j+1}{1}^{p_1}\binom{j+1}{2}^{p_2}\binom{j+1}{3}^{p_3}\ldots,$$

with the simplest cases

$$\binom{j+2}{2}=\binom{j+1}{1}^2-\binom{j+1}{2},$$

$$\binom{j+3}{3}=\binom{j+1}{1}^3-2\binom{j+1}{2}\binom{j+1}{1}+\binom{j+1}{3}.$$

317. (vi) We have

$$x^{\frac{1}{2}i(i-1)}\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{i}\end{matrix}\right]=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p)!}{p_1!\,p_2!\,p_3!\ldots}\left[\begin{matrix}\mathbf{j+1}\\ \mathbf{1}\end{matrix}\right]^{p_1}\left[\begin{matrix}\mathbf{j+2}\\ \mathbf{2}\end{matrix}\right]^{p_2}\left[\begin{matrix}\mathbf{j+3}\\ \mathbf{3}\end{matrix}\right]^{p_3}\ldots.$$

Putting x equal to unity we have the arithmetical result

$$\binom{j+1}{i}=\Sigma(-)^{i+\Sigma p}\frac{(\Sigma p)!}{p_1!\,p_2!\,p_3!\ldots}\binom{j+1}{1}^{p_1}\binom{j+2}{2}^{p_2}\binom{j+3}{3}^{p_3}\ldots,$$

with the simplest cases

$$\binom{j+1}{2}=\binom{j+1}{1}^2-\binom{j+2}{2},$$

$$\binom{j+1}{3}=\binom{j+1}{1}^3-2\binom{j+2}{2}\binom{j+1}{1}+\binom{j+3}{3}.$$

318. It is to be observed that these several transformations are not only intermediate steps but are in themselves noteworthy theorems in partitions. Each of the partial fractions that is obtained is in itself an enumerating generating function in the theory. This is not the case in the ordinary

method of partial fractions that has been developed in this connexion by Cayley and Sylvester.

The breaking up into partial fractions of the fraction

$$\frac{(\mathrm{j}+1)(\mathrm{j}+2)\ldots(\mathrm{j}+\mathrm{i})}{(1)(2)\ldots(\mathrm{i})}$$

has one important property that must not be overlooked.

The function as it stands is not visibly a finite integral function and in order to express it as a product of finite integral functions it is necessary to know the actual values of i and j so as to find the irreducible factor of the various numerator and denominator factors. In the transformations before us in this chapter the partial fractions which present themselves are such that each is visibly a product of finite integral functions and consequently itself a finite integral function.

319. Speaking in general we may in any relation which connects the quantities s_i, a_i, h_i substitute either

(i) $$s_i = \frac{1}{(\mathrm{i})},$$

$$a_i = \frac{x^{\frac{1}{2}i(i-1)}}{(1)(2)\ldots(\mathrm{i})}.$$

$$h_i = \frac{1}{(1)(2)\ldots(\mathrm{i})};$$

or (ii) $$s_i = \frac{(\mathrm{ij}+\mathrm{i})}{(\mathrm{i})},$$

$$a_i = x^{\frac{1}{2}i(i-1)}\frac{(\mathrm{j}-\mathrm{i}+2)(\mathrm{j}-\mathrm{i}+3)\ldots(\mathrm{j}+1)}{(1)(2)\ldots(\mathrm{i})},$$

$$h_i = \frac{(\mathrm{j}+1)(\mathrm{j}+2)\ldots(\mathrm{j}+\mathrm{i})}{(1)(2)\ldots(\mathrm{i})};$$

and since each of these expressions is an enumerating generating function in partition theory it follows that every such relation involves a theorem in partitions.

320. In particular the relations between the quantities a_i, h_i are interesting because in any such relation, as we have seen in an earlier section of this work, we may interchange the symbols a, h.

Moreover there is an infinite number of functions of the quantities a such that each function remains unaltered when h is written for a. In fact s_{2p+1} when expressed in terms of a_1, a_2, a_3, ... has this property and s_{2p} merely changes sign.

In order therefore to form such functions we may take any product

$$s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4} \ldots,$$

such that

$$p_2 + p_4 + p_6 + \ldots$$

is even.

Every linear function of such products has the desired property.

As one of the simplest examples we have

$$\tfrac{1}{3}(s_1^3 - s_3) = a_1 a_2 - a_3,$$

leading us to the relation

$$a_1 a_2 - a_3 = h_1 h_2 - h_3,$$

and thence to

$$\frac{x}{(1)^2 (2)} - \frac{x^3}{(1)(2)(3)} = \frac{1}{(1)^2 (2)} - \frac{1}{(1)(2)(3)},$$

which we may write in Cayley's Notation

$$P(1, 1, 2)(w-1) - P(1, 2, 3)(w-3)$$
$$= P(1, 1, 2)\, w - P(1, 2, 3)\, w,$$

or in another form

$$P(1, 2, 3)\, w - P(1, 2, 3)(w-3)$$
$$= P(1, 1, 2)\, w - P(1, 1, 2)(w-1),$$

a relation which, in this simple case, is obviously true because each side is equal to

$$P(1, 2)\, w.$$

321. Again taking the second set of expressions for s_i, a_i, h_i we have

$$x \frac{(j)(j+1)^2}{(1)^2 (2)} - x^3 \frac{(j-1)(j)(j+1)}{(1)(2)(3)}$$
$$= \frac{(j+1)^2 (j+2)}{(1)^2 (2)} - \frac{(j+1)(j+2)(j+3)}{(1)(2)(3)},$$

a relation which is not immediately obvious.

It leads us to

$$\sum_s P(1\,;\, j)(w-s)\, P(1, 2\,;\, j-1)(s-1) - P(1, 2, 3\,;\, j-1)(w-3)$$
$$= \sum_s P(1\,;\, j)(w-s)\, P(1, 2\,;\, j)\, s - P(1, 2, 3\,;\, j)\, w.$$

322. The method of partial fractions and other methods have been applied to the actual evaluation of the coefficients in the expansions of enumerating functions in the theory of partitions. The reader is referred to the original researches of Cayley* and Sylvester†, and to a very good short summary in Netto's *Combinatorik*‡.

* *Coll. Math. Papers*, Vol. II., pp. 235, 506.

† *Math. Papers*, Vol. II., p. 90.

‡ *Lehrbuch der Combinatorik*, Leipzig, 1901, p. 146 *et seq.*

323. The function

$$\frac{(j+1)(j+2)\dots(j+i)}{(1)(2)\dots(i)}$$

is intimately connected with the limited double product

$$(1+ax)(1+ax^3)\dots(1+ax^{2i-1})$$

into

$$\left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\dots\left(1+\frac{x^{2j-1}}{a}\right).$$

Mathematicians have almost invariably considered the case of the double product in which $j=i$ and have thus not directly connected it with the function which we are studying.

Sylvester only considers the special case, but his quasi-geometrical method of demonstration is equally available for the general case and is the one to which attention will now be drawn.

The coefficient $a^m x^w$ in the first part of the product denotes the number of ways of composing w with m unrepeated uneven numbers none of which is greater than $2i-1$. Similarly in the second part the coefficient of $a^{-m}x^w$ gives the number of ways of composing w with m unrepeated uneven numbers none of which is greater than $2j-1$. The product is not symmetrical in a and $\frac{1}{a}$ unless j is equal to i, but observe that it is unchanged by the simultaneous substitution of $\frac{1}{a}$ for a and interchange of i and j. It results from this remark that the coefficients of a^m and a^{-m} in the double product are derivable the one from the other by merely interchanging i and j. The coefficient of a^m in the double product is obtained by multiplying the coefficient of $a^{m+\theta}$ in the first part of the product by the coefficient of $a^{-\theta}$ in the second part, giving θ all values from zero to j (so long as $m+\theta$ does not exceed i) and adding the results. The coefficient of $a^{m+\theta}$ in the first part is a function of x which enumerates the partitions of numbers into $m+\theta$ unrepeated uneven parts none of which is greater than $2i-1$. In this function of x the power of x, viz. x^{w_1}, occurs with a coefficient C_{w_1} indicating that there are C_{w_1} such partitions of the number w_1. Let one of these partitions be

$$(l_1, l_2, \dots l_m, l_{m+1}, \dots l_{m+\theta}),$$

the parts being, as usual, written in descending order of magnitude. Also, in the second part the coefficient of $a^{-\theta}x^{w_2}$ gives the number of ways of partitioning the number w_2 into θ unrepeated uneven parts none of which is greater than $2j-1$. Let one of these partitions be

$$(\lambda_1, \lambda_2, \dots \lambda_\theta),$$

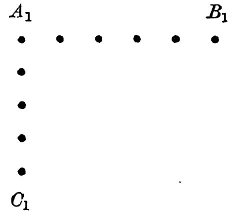
the parts being written in descending order of magnitude.

We now form a graph from the two sets of numbers

$$l_{m+1}, \quad l_{m+2}, \ \ldots \ l_{m+\theta},$$
$$\lambda_1, \quad \lambda_2, \ \ldots \ \lambda_\theta,$$

in the following manner.

Calling i the major half of the uneven number $2i-1$, we form an angle of nodes



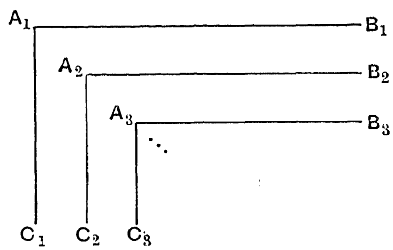
where the number of nodes in the line A_1B_1 is the major half of l_{m+1} and the number of nodes in the line A_1C_1 is the major half of λ_1. The number of nodes in the angle is thus

$$\tfrac{1}{2}(l_{m+1}+\lambda_1).$$

Similarly we form angles of nodes $B_2A_2C_2$, $B_3A_3C_3$, ... $B_\theta A_\theta C_\theta$ involving numbers of nodes equal to

$$\tfrac{1}{2}(l_{m+2}+\lambda_2), \ \tfrac{1}{2}(l_{m+3}+\lambda_3), \ \ldots \ \tfrac{1}{2}(l_{m+\theta}+\lambda_\theta)$$

respectively, and then fit the angles together in the manner



so as to form a regular graph of content

$$\tfrac{1}{2}(l_{m+1}+l_{m+2}+\ldots+l_{m+\theta}+\lambda_1+\lambda_2+\ldots+\lambda_\theta).$$

The number of nodes in the first row of the graph is

$$\tfrac{1}{2}(l_{m+1}+1),$$

and the number in the first column is

$$\tfrac{1}{2}(\lambda_1+1).$$

Since $\qquad l_1 \not> 2i-1, \ l_2 \not> 2i-3, \ \ldots \ l_{m+1} \not> 2i-2m-1,$

$$\lambda_1 \not> 2j-1,$$

it is clear that the first row and the first column are limited to contain not more than $i-m$ and j nodes respectively.

Now form a second graph of m rows containing

$$\begin{gathered}\tfrac{1}{2}(l_1 - 2m + 1),\\ \tfrac{1}{2}(l_2 - 2m + 3),\\ \vdots\\ \tfrac{1}{2}(l_{m-1} - 3),\\ \tfrac{1}{2}(l_m - 1)\end{gathered}$$

nodes respectively.

The content of this graph is

$$\tfrac{1}{2}(l_1 + l_2 + \ldots + l_m - m^2),$$

and the rows are limited to contain not more than $i - m$ nodes.

If we now superpose this graph to the former (as we may, because $\frac{1}{2}l_m - \frac{1}{2} \geqslant \frac{1}{2}l_{m+1} + \frac{1}{2}$) we obtain a graph of content

$$\tfrac{1}{2}(l_1 + l_2 + \ldots + l_{m+\theta} + \lambda_1 + \lambda_2 + \ldots + \lambda_\theta - m^2),$$

which is

$$\tfrac{1}{2}(w_1 + w_2 - m^2)$$

or

$$\tfrac{1}{2}(w - m^2) \quad \text{where} \quad w = w_1 + w_2.$$

The number of nodes in the first row is

$$\tfrac{1}{2}(l_1 - 2m + 1),$$

and in the first column

$$m + \tfrac{1}{2}(\lambda_1 + 1),$$

numbers which are limited not to exceed

$$i - m, \; j + m \quad \text{respectively.}$$

The graph that has been constructed is specified by the content $\frac{1}{2}(w - m^2)$ and by the two limiting numbers $i - m, j + m$.

The number θ does not enter into the specification.

By varying θ and the two sets of numbers

$$l_1, l_2, \ldots l_{m+\theta}; \; \lambda_1, \lambda_2, \ldots \lambda_\theta$$

we arrive at all graphs which have the defined specification.

This is so because from any graph so specified the numbers

$$\theta; \; l_1, l_2, \ldots l_{m+\theta}; \; \lambda_1, \lambda_2, \ldots \lambda_\theta$$

can be recovered for a given value of the integer m.

It follows that the coefficient of $a^m x^w$ in the double product is equal to the coefficient of $x^{\frac{1}{2}(w - m^2)}$ in the function which enumerates partitions limited by the two numbers

$$i - m, \quad j + m.$$

This function is

$$\frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i-m).(1)(2)\ldots(j+m)}.$$

The coefficient herein of $x^{\frac{1}{2}(w-m^2)}$ is equal to the coefficient of x^w in

$$\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i-2m).(2)(4)\ldots(2j+2m)}x^{m^2}.$$

Hence the coefficients of a^m and a^{-m} in the double product are

$$\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i-2m).(2)(4)\ldots(2j+2m)}x^{m^2},$$

$$\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i+2m).(2)(4)\ldots(2j-2m)}x^{m^2},$$

respectively.

We have thus the identity

$$(1+ax)(1+ax^3)\ldots(1+ax^{2i-1})$$
$$\times\left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\ldots\left(1+\frac{x^{2j-1}}{a}\right)$$
$$=\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i).(2)(4)\ldots(2j)}$$
$$+\sum_m a^m\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i-2m).(2)(4)\ldots(2j+2m)}x^{m^2}$$
$$+\sum_m a^{-m}\frac{(2)(4)\ldots(2i+2j)}{(2)(4)\ldots(2i+2m).(2)(4)\ldots(2j-2m)}x^{m^2}.$$

324. We now make an important transformation.

Observing that if we write ax for a, each side of the identity becomes a function of x^2, we so write and subsequently write x for x^2.

We thus obtain the identity

$$(1+ax)(1+ax^2)(1+ax^3)\ldots(1+ax^i)$$
$$\times\left(1+\frac{1}{a}\right)\left(1+\frac{x}{a}\right)\left(1+\frac{x^2}{a}\right)\ldots\left(1+\frac{x^{j-1}}{a}\right)$$
$$=\frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i).(1)(2)\ldots(j)}$$
$$+\sum a^m\frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i-m).(1)(2)\ldots(j+m)}x^{\frac{1}{2}m(m+1)}$$
$$+\sum a^{-m}\frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i+m).(1)(2)\ldots(j-m)}x^{\frac{1}{2}m(m-1)}$$

$$= \frac{(1)(2)\dots(i+j)}{(1)(2)\dots(i).(1)(2)\dots(j)}$$

$$\times \left\{1 + \frac{(i)}{(j+1)} xa + \frac{(j)}{(i+1)} \frac{1}{a}\right.$$

$$+ \frac{(i-1)(i)}{(j+1)(j+2)} x^3 a^2 + \frac{(j-1)(j)}{(i+1)(i+2)} \frac{x}{a^2}$$

$$+ \frac{(i-2)(i-1)(i)}{(j+1)(j+2)(j+3)} x^6 a^3 + \frac{(j-2)(j-1)(j)}{(i+1)(i+2)(i+3)} \frac{x^3}{a^3}$$

$$\left. + \dots\dots\dots\dots\dots\dots\dots\dots\dots \right\}^*.$$

325. The limited double product may be dealt with in another manner which will lead to an important identity.

For

$$(1+ax)(1+ax^3)\dots(1+ax^{2i-1}) \times \left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\dots\left(1+\frac{x^{2j-1}}{a}\right)$$

$$= 1 + ax\frac{(2i)}{(2)} + a^2x^4 \frac{(2i-2)(2i)}{(2)(4)} + a^3x^9 \frac{(2i-4)(2i-2)(2i)}{(2)(4)(6)} + \dots,$$

into $$1 + \frac{x}{a}\frac{(2j)}{(2)} + \frac{x^4}{a^2}\frac{(2j-2)(2j)}{(2)(4)} + \frac{x^9}{a^3}\frac{(2j-4)(2j-2)(2j)}{(2)(4)(6)} + \dots,$$

and by simple multiplication

$$= 1 + x^2 \frac{(2i)(2j)}{(2)^2} + x^8 \frac{(2i-2)(2i).(2j-2)(2j)}{(2)^2(4)^2} + \dots$$

$$+ \Sigma a^m \left\{ x^{m^2} \frac{(2i-2m+2)\dots(2i)}{(2)(4)\dots(2m)} + x^{(m+1)^2+1^2} \frac{(2i-2m)\dots(2i)}{(2)(4)\dots(2m+2)} \cdot \frac{(2j)}{(2)}\right.$$

$$+ x^{(m+2)^2+2^2} \frac{(2i-2m-2)\dots(2i)}{(2)(4)\dots(2m+4)} \cdot \frac{(2j-2)(2j)}{(2)(4)}$$

$$+ x^{(m+3)^2+3^2} \frac{(2i-2m-4)\dots(2i)}{(2)(4)\dots(2m+6)} \cdot \frac{(2j-4)(2j-2)(2j)}{(2)(4)(6)}$$

$$\left. + \dots\dots\dots\dots\dots\dots\dots\dots\dots \right\}$$

$$+ \Sigma a^{-m} \left\{ x^{m^2} \frac{(2j-2m+2)\dots(2j)}{(2)(4)\dots(2m)} + x^{(m+1)^2+1^2} \frac{(2j-2m)\dots(2j)}{(2)(4)\dots(2m+2)} \cdot \frac{(2i)}{(2)}\right.$$

$$+ x^{(m+2)^2+2^2} \frac{(2j-2m-2)\dots(2j)}{(2)(4)\dots(2m+4)} \cdot \frac{(2i-2)(2i)}{(2)(4)}$$

$$\left. + \dots\dots\dots\dots\dots\dots\dots\dots\dots \right\}.$$

* See Sylvester, *Collected Mathematical Papers*, Vol. IV., "A Constructive Theory of Partitions," where the particular case $j=i$ is considered.

Now put ax for x, and subsequently write x for x^2, obtaining

$$(1+ax)(1+ax^2)\dots(1+ax^i)\times\left(1+\frac{1}{a}\right)\left(1+\frac{x}{a}\right)\dots\left(1+\frac{x^{j-1}}{a}\right)$$

$$=1+x\frac{(\mathbf{i})(\mathbf{j})}{(\mathbf{1})^2}+x^4\frac{(\mathbf{i}-\mathbf{1})(\mathbf{i}).(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})^2(\mathbf{2})^2}+\dots$$

$$+\Sigma a^m\left\{x^{\frac{1}{2}m(m+1)}\frac{(\mathbf{i}-\mathbf{m}+\mathbf{1})\dots(\mathbf{i})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m})}+x^{\frac{1}{2}(m^2+3m+2)}\frac{(\mathbf{i}-\mathbf{m})(\mathbf{i}-\mathbf{m}+\mathbf{1})\dots(\mathbf{i})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{1})}\cdot\frac{(\mathbf{j})}{(\mathbf{1})}\right.$$

$$+x^{\frac{1}{2}(m^2+5m+8)}\frac{(\mathbf{i}-\mathbf{m}-\mathbf{1})(\mathbf{i}-\mathbf{m})\dots(\mathbf{i})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{2})}\cdot\frac{(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})(\mathbf{2})}$$

$$+x^{\frac{1}{2}(m^2+7m+18)}\frac{(\mathbf{i}-\mathbf{m}-\mathbf{2})(\mathbf{i}-\mathbf{m}-\mathbf{1})\dots(\mathbf{i})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{3})}\cdot\frac{(\mathbf{j}-\mathbf{2})(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}$$

$$\left.+\dots\dots\dots\dots\dots\dots\right\}$$

$$+\Sigma a^{-m}\left\{x^{\frac{1}{2}m(m-1)}\frac{(\mathbf{j}-\mathbf{m}+\mathbf{1})\dots(\mathbf{j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m})}+x^{\frac{1}{2}(m^2+m+2)}\frac{(\mathbf{j}-\mathbf{m})(\mathbf{j}-\mathbf{m}+\mathbf{1})\dots(\mathbf{j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{1})}\cdot\frac{(\mathbf{i})}{(\mathbf{1})}\right.$$

$$+x^{\frac{1}{2}(m^2+3m+8)}\frac{(\mathbf{j}-\mathbf{m}-\mathbf{1})(\mathbf{j}-\mathbf{m})\dots(\mathbf{j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{2})}\cdot\frac{(\mathbf{i}-\mathbf{1})(\mathbf{i})}{(\mathbf{1})(\mathbf{2})}$$

$$+x^{\frac{1}{2}(m^2+5m+18)}\frac{(\mathbf{j}-\mathbf{m}-\mathbf{2})(\mathbf{j}-\mathbf{m}-\mathbf{1})\dots(\mathbf{j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{m}+\mathbf{3})}\cdot\frac{(\mathbf{i}-\mathbf{2})(\mathbf{i}-\mathbf{1})(\mathbf{i})}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}$$

$$\left.+\dots\dots\dots\dots\dots\dots\right\}.$$

326. Now looking to the prior result and equating coefficients of like powers of a we find, after striking out common factors, the identities

$$\text{(i)}\quad 1+x\frac{(\mathbf{i})(\mathbf{j})}{(\mathbf{1})^2}+x^4\frac{(\mathbf{i}-\mathbf{1})(\mathbf{i}).(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})^2(\mathbf{2})^2}+x^9\frac{(\mathbf{i}-\mathbf{2})\dots(\mathbf{i}).(\mathbf{j}-\mathbf{2})\dots(\mathbf{j})}{(\mathbf{1})^2(\mathbf{2})^2(\mathbf{3})^2}+\dots$$

$$=\frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}+\mathbf{j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}).(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})},$$

which is hereafter reached from a consideration of Permutation Functions;

$$\text{(ii)}\quad 1+x^{m+1}\frac{(\mathbf{i}-\mathbf{m})}{(\mathbf{m}+\mathbf{1})}\cdot\frac{(\mathbf{j})}{(\mathbf{1})}+x^{2m+4}\frac{(\mathbf{i}-\mathbf{m}-\mathbf{1})(\mathbf{i}-\mathbf{m})}{(\mathbf{m}+\mathbf{1})(\mathbf{m}+\mathbf{2})}\cdot\frac{(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})(\mathbf{2})}$$

$$+x^{3m+9}\frac{(\mathbf{i}-\mathbf{m}-\mathbf{2})(\mathbf{i}-\mathbf{m}-\mathbf{1})(\mathbf{i}-\mathbf{m})}{(\mathbf{m}+\mathbf{1})(\mathbf{m}+\mathbf{2})(\mathbf{m}+\mathbf{3})}\cdot\frac{(\mathbf{j}-\mathbf{2})(\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})(\mathbf{2})(\mathbf{3})}$$

$$+\dots\dots\dots\dots\dots\dots$$

$$=\frac{(\mathbf{j}+\mathbf{m}+\mathbf{1})(\mathbf{j}+\mathbf{m}+\mathbf{2})\dots(\mathbf{j}+\mathbf{i})}{(\mathbf{m}+\mathbf{1})(\mathbf{m}+\mathbf{2})\dots(\mathbf{i})},$$

derived by comparison of the coefficients of a^m.

Comparison of the coefficients of a^{-m} gives the same formula with the mere interchange of i and j.

327. In the concluding articles of this chapter it is proposed to give some of the methods and processes of Gauss in the study of partition series. They are highly ingenious and no student would find his equipment complete without them. The notation is to a large extent altered to be in accordance with that used in this work.

Consider the product of two infinite products

$$(1+ax)(1+ax^3)(1+ax^5)\dots$$
$$\times\left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\left(1+\frac{x^5}{a}\right)\dots,$$

and suppose multiplication to produce

$$\dots+\frac{R}{a^2}+\frac{Q}{a}+P+Qa+Ra^2+\dots.$$

Writing ax^2 for a has the effect of multiplying the product by

$$\frac{1+\dfrac{1}{ax}}{1+ax}=\frac{1}{ax},$$

hence

$$(1+ax^3)(1+ax^5)\dots\left(1+\frac{1}{ax}\right)\left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\left(1+\frac{x^5}{a}\right)\dots$$
$$=\dots+\frac{R}{a^2x^4}+\frac{Q}{ax^2}+P+Qax^2+Ra^2x^4+\dots$$
$$=\dots+\frac{R}{a^3x}+\frac{Q}{a^2x}+\frac{P}{ax}+\frac{Q}{x}+R\frac{a}{x}+\dots,$$

the last line but one being derived by writing ax^2 for a, and the last line by multiplication by $\dfrac{1}{ax}$.

Comparison gives $\qquad \dots=R=Qx^3=Px^4=\dots,$

and thence the right-hand side

$$\dots+\frac{R}{a^2}+\frac{Q}{a}+P+Qa+Ra^2+\dots$$

of the above assumed identity may be written

$$P\left\{1+x\left(a+\frac{1}{a}\right)+x^4\left(a^2+\frac{1}{a^2}\right)+x^9\left(a^3+\frac{1}{a^3}\right)+\dots\right\},$$

and it only remains to calculate P which is the term independent of a in the given product.

328. He shews by precisely the same method that if the square of last considered product be written

$$\ldots+\frac{Q}{a}+P+Qa+\ldots,$$

then

$$(1+ax)^2(1+ax^3)^2(1+ax^5)^2\ldots\left(1+\frac{x}{a}\right)^2\left(1+\frac{x^3}{a}\right)^2\left(1+\frac{x^5}{a}\right)^2\ldots$$

$$=P\left\{1+x^2\left(a^2+\frac{1}{a^2}\right)+x^8\left(a^4+\frac{1}{a^4}\right)+x^{18}\left(a^6+\frac{1}{a^6}\right)+\ldots\right\}$$

$$+Q\left\{\left(a+\frac{1}{a}\right)+x^4\left(a^3+\frac{1}{a^3}\right)+x^{12}\left(a^5+\frac{1}{a^5}\right)+x^{24}\left(a^7+\frac{1}{a^7}\right)+\ldots\right\},$$

the exponents of x in the P series being the doubles of the square numbers and in the Q series the quadruples of the figurate numbers of the third order.

329. Write

$$P=1+\frac{x^n}{1+x^n}\cdot\frac{1-x^{2n+1}}{1+x^{n+1}}+\frac{x^{2n}}{1+x^n}\cdot\frac{1-x^{2n+2}}{1+x^{n+1}}\cdot\frac{1-x^{n+1}}{1+x^{n+2}}$$

$$+\frac{x^{3n}}{1+x^n}\cdot\frac{1-x^{2n+3}}{1+x^{n+1}}\cdot\frac{1-x^{n+1}}{1+x^{n+2}}\cdot\frac{1-x^{n+2}}{1+x^{n+3}}+\ldots,$$

$$Q=\frac{x^n}{1+x^n}+\frac{x^{2n}}{1+x^n}\cdot\frac{1-x^{n+1}}{1+x^{n+1}}+\frac{x^{3n}}{1+x^n}\,\frac{1-x^{n+1}}{1+x^{n+1}}\cdot\frac{1-x^{n+2}}{1+x^{n+2}}+\ldots,$$

$$R=P-Q.$$

We first of all calculate R by subtracting the terms of Q from those of P in order. We find

$$R=\frac{1}{1+x^n}+\frac{x^n(1-x^n)}{(1+x^n)(1+x^{n+1})}+\frac{x^{2n}(1-x^n)(1-x^{n+1})}{(1+x^n)(1+x^{n+1})(1+x^{n+2})}+\ldots=\phi(x,n).$$

But if we find R by adding the first term of P, the difference between the second term of P and the first of Q, between the third term of P and the second of Q and so forth, we find

$$R=1-\frac{x^{2n+1}}{1+x^{n+1}}-\frac{x^{3n+2}(1-x^{n+1})}{(1+x^{n+1})(1+x^{n+2})}-\frac{x^{4n+3}(1-x^{n+1})(1-x^{n+2})}{(1+x^{n+1})(1+x^{n+2})(1+x^{n+3})}-\ldots$$

$$=1-x^{2n+1}\phi(x,n+1),$$

or

$$\phi(x,n)=1-x^{2n+1}\phi(x,n+1),$$

true when n is an integer greater than unity; and with this restriction we are led to the identity

$$\phi(x,n)=1-x^{2n+1}+x^{4n+4}-x^{6n+9}+x^{8n+16}-\ldots.$$

When $n=0$, the last term of Q must be taken into account. Calling it Q' the second series for R is greater than the first series for R by Q'. Thus

$$1 - x\phi(x, 1) = \phi(x, 0) + Q',$$

leading to

$$\phi(x, 0) = 1 - x\phi(x, 1) - Q' = 1 - x + x^4 - x^9 + x^{16} - \ldots - Q'.$$

Now $\phi(x, 0) = \frac{1}{2}$ by putting $n=0$ in the first series for R, and from the series for Q

$$Q' = \frac{1}{2}\frac{(1-x)(1-x^2)(1-x^3)\ldots}{(1+x)(1+x^2)(1+x^3)\ldots}.$$

Hence

$$\frac{(1-x)(1-x^2)(1-x^3)\ldots}{(1+x)(1+x^2)(1+x^3)\ldots} = 1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \ldots,$$

an important result first obtained from elliptic function theory.

The left-hand side of the identity may be written

$$(1)^2 (2) (3)^2 (4) (5)^2 (6) \ldots,$$

so that writing $\qquad 1 - 2x + 2x^4 - 2x^9 + \ldots = F(x),$

we find $\qquad F(x) F(-x) = \{F(x^2)\}^2.$

330. In a similar manner by considering the two series

$$\frac{1-x^{2n+2}}{1-x^{n+1}} + \frac{x^n(1-x^{2n+4})(1-x^{n+2})}{(1-x^{n+1})(1-x^{n+3})} + \frac{x^{2n}(1-x^{2n+6})(1-x^{n+2})(1-x^{n+4})}{(1-x^{n+1})(1-x^{n+3})(1-x^{n+5})} + \ldots,$$

$$\frac{x^n(1-x^{n+2})}{1-x^{n+1}} + \frac{x^{2n}(1-x^{n+2})(1-x^{n+4})}{(1-x^{n+1})(1-x^{n+3})} + \frac{x^{3n}(1-x^{n+2})(1-x^{n+4})(1-x^{n+6})}{(1-x^{n+1})(1-x^{n+3})(1-x^{n+5})} + \ldots$$

it is shewn that

$$\frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1-x)(1-x^3)(1-x^5)\ldots} = 1 + x + x^3 + x^6 + x^{10} + \ldots,$$

and thence by changing the sign of x

$$\frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1+x)(1+x^3)(1+x^5)\ldots} = 1 - x - x^3 + x^6 + x^{10} - \ldots.$$

Putting $\qquad 1 - x - x^3 + x^6 + x^{10} - \ldots = \phi(x),$

and as before $\qquad 1 - 2x + 2x^4 - 2x^9 + \ldots = F(x),$

it is easy to shew that

$$\phi(x)\phi(-x) = \phi(x^2) F(x^4).$$

331. In the result of Art. 327 if we put $a=1$, comparison with the expression that has been obtained for

$$1-2x+2x^4-2x^9+\dots$$

shews that

$$(1+ax)(1+ax^3)(1+ax^5)\dots\left(1+\frac{x}{a}\right)\left(1+\frac{x^3}{a}\right)\left(1+\frac{x^5}{a}\right)\dots$$

$$=\frac{1+(a+a^{-1})x+(a^2+a^{-2})x^4+(a^3+a^{-3})x^9+\dots}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)\dots}.$$

If herein we write x^3 for x and $-x$ for a we find

$$(4)(10)(16)\dots(2)(8)(14)\dots$$

$$=\frac{1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+\dots}{(6)(12)(18)(24)\dots},$$

or multiplying up and writing x for x^2 we get

$$(1)(2)(3)(4)\dots=1-x-x^2+x^5+x^7-x^{12}-x^{15}+\dots,$$

an interesting way of arriving at this celebrated identity.

If on the other hand we write x^3 for x and $+x$ for a we find

$$1+x+x^2+x^5+x^7+x^{12}+x^{15}+\dots=\frac{(3)(6)(9)(12)(15)(18)\dots}{(1)(5)(7)(11)(13)(17)(19)\dots}.$$

332. Following Gauss we denote the function

$$\frac{(j+1)(j+2)\dots(j+i)}{(1)(2)\dots(i)} \text{ by } (j+i,\, i) \text{ or } (j+i,\, j).$$

Since $\quad (i+j,\, i+1)=(i+j-1,\, i+1)+x^{j-1}(i+j-1,\, i),$

$$(i+j-1,\, i+1)=(i+j-2,\, i+1)+x^{j-2}(i+j-2,\, i),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(i+2,\, i+1)=1+x(i+1,\, i),$$

it is easy to shew that

$$(i+j,\, i+1)=1+x(i+1,\, i)+x^2(i+2,\, i)+\dots+x^{j-1}(i+j-1,\, i).$$

Now consider the series

$$f(x,j)=1-\frac{1-x^{j+1}}{1-x}+\frac{(1-x^j)(1-x^{j+1})}{(1-x)(1-x^2)}-\frac{(1-x^{j-1})(1-x^j)(1-x^{j+1})}{(1-x)(1-x^2)(1-x^3)}+\dots,$$

j being of course a positive integer. The series has $j+2$ terms, the last term being ± 1 according as j is uneven or even.

Writing

$$f(x, j) = 1 - (j+1, 1) + (j+1, 2) - (j+1, 3) + \dots$$

we have

$$1 = 1,$$
$$-(j+1, 1) = -(j, 1) - x^j,$$
$$+(j+1, 2) = +(j, 2) + x^{j-1}(j, 1),$$
$$-(j+1, 3) = -(j, 3) - x^{j-2}(j, 2),$$
etc.,

leading to

$$f(x, j) = (1 - x^j) - (1 - x^{j-1})(j, 1) + (1 - x^{j-2})(j, 2) - \dots,$$

but

$$(1 - x^{j-s})(j, s) = (1 - x^j)(j-1, s),$$

so that

$$f(x, j) = (1 - x^j) f(x, j-2).$$

Now

$$f(x, 0) = 0, \quad f(x, 1) = 1 - x, \quad f(x, 2) = 0,$$

and we find

$$f(x, 1) = 1 - x,$$
$$f(x, 3) = (1 - x)(1 - x^3),$$
$$f(x, 5) = (1 - x)(1 - x^3)(1 - x^5),$$
etc.,

and when j is uneven

$$f(x, j) = (1 - x)(1 - x^3)(1 - x^5) \dots (1 - x^j).$$

Clearly, when j is even, $\qquad f(x, j) = 0,$

and it is manifest that the terms destroy one another in pairs.

333. Next consider the series

$$F(x, j) = 1 + x^{\frac{1}{2}} \frac{1 - x^{j+1}}{1 - x} + x \frac{(1 - x^j)(1 - x^{j+1})}{(1 - x)(1 - x^2)}$$
$$+ x^{\frac{3}{2}} \frac{(1 - x^{j-1})(1 - x^j)(1 - x^{j+1})}{(1 - x)(1 - x^2)(1 - x^3)} + \dots,$$

or

$$1 + x^{\frac{1}{2}}(j+1, 1) + x(j+1, 2) + x^{\frac{3}{2}}(j+1, 3) + \dots.$$

Reversing the series, it may be written

$$F(x, j) = x^{\frac{1}{2}(j+1)} + x^{\frac{1}{2}j}(j+1, 1) + x^{\frac{1}{2}(j-1)}(j+1, 2) + x^{\frac{1}{2}(j-2)}(j+1, 3) + \dots.$$

Multiply this series by $x^{\frac{1}{2}(j+2)}$ and add it, term to term, to the first series, obtaining

$$\{1 + x^{\frac{1}{2}(j+2)}\} F(x, j)$$
$$= 1 + x^{\frac{1}{2}}(j+1, 1) + x(j+1, 2) + x^{\frac{3}{2}}(j+1, 3) + x^2(j+1, 4) + \dots$$
$$+ x^{\frac{1}{2}} . x^{j+1} + x . x^j (j+1, 1) + x^{\frac{3}{2}} . x^{j-1}(j+1, 2) + x^2 . x^{j-2}(j+1, 3) + \dots.$$

Now by the last article

$$(j+1,\,1) + x^{j+1} = (j+2,\,1),$$
$$(j+1,\,2) + x^{j}(j+1,\,1) = (j+2,\,2),$$
$$(j+1,\,3) + x^{j-1}(j+1,\,2) = (j+2,\,3),$$
$$\text{etc.},$$

so that
$$\{1 + x^{\frac{1}{2}(j+2)}\}\,F(x, j) = F(x, j+1).$$

Moreover
$$F(x, -1) = 0,$$

so that
$$F(x, 0) = 1 + x^{\frac{1}{2}},$$
$$F(x, 1) = (1 + x^{\frac{1}{2}})(1 + x),$$
$$F(x, 2) = (1 + x^{\frac{1}{2}})(1 + x)(1 + x^{\frac{3}{2}}),$$

and in general
$$F(x, j) = (1 + x^{\frac{1}{2}})(1 + x)(1 + x^{\frac{3}{2}}) \ldots (1 + x^{\frac{1}{2}j + \frac{1}{2}}).$$

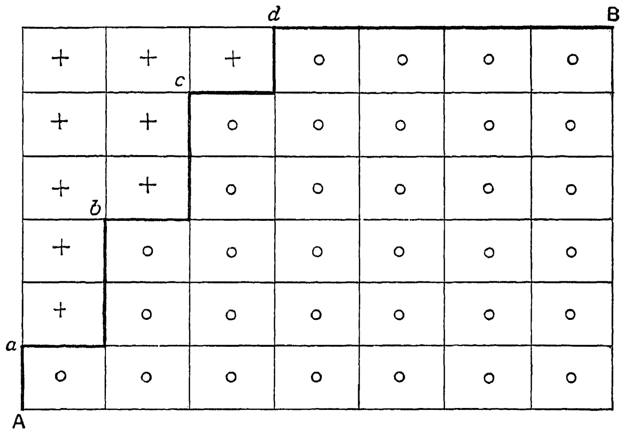
Writing x^2 for x we have

$$1 + x\frac{(2j+2)}{(2)} + x^2\frac{(2j)\,(2j+2)}{(2)\,(4)} + x^3\frac{(2j-2)\,(2j)\,(2j+2)}{(2)\,(4)\,(6)} + \ldots$$
$$= (1+x)(1+x^2)(1+x^3)\ldots(1+x^{j+1}).$$

CHAPTER VI

CONNEXION OF THE THEORY OF PARTITIONS WITH OTHER COMBINATORY THEORIES

334. It has been shewn that the partitions of unipartite numbers into parts limited as to magnitude by p and as to number by q can be graphically represented by nodes or units placed in the cells of a rectangular lattice formed by cutting $p+1$ vertical lines by $q+1$ horizontal lines.



As a particular and representative case take $p=7$, $q=6$ and construct the lattice. On it depict by crosses the unipartite partition 32211 of the number 9. If the unoccupied cells be filled by noughts we obtain a complementary partition 766554 of the number 33, where

$$9+33=42=7\times 6=pq.$$

Now separate these two partitions by the path marked in black.

The whole lattice denotes, by Sect. IV, Ch. II, the graph of the bipartite number $\overline{pq}$ and the blackened path is a line of route through it. This line of

route denotes a principal composition of the bipartite $\overline{pq}$ and also by Sect. IV, Ch. I, it is the zig-zag graph of a composition of the unipartite number

$$p+q+1.$$

The line of route also denotes a permutation of the letters

$$\alpha^p \beta^q.$$

335. It follows that partitions of unipartite numbers are closely connected with

(i) the compositions of bipartite numbers;

(ii) the compositions of unipartite numbers;

(iii) permutations involving two different letters.

In the above diagram we have

(i) the composition $(\overline{01}\ \overline{12}\ \overline{12}\ \overline{11}\ \overline{40})$ of the bipartite $\overline{76}$,

(ii) the composition 23321111 of the unipartite 14,

(iii) the permutation $\beta\alpha\beta\beta\alpha\beta\beta\alpha\beta\alpha\alpha\alpha\alpha$ of $\alpha^7\beta^6$.

The compositions of the bipartite number $\overline{pq}$ which are under examination are those in which consecutive figures of successive biparts are both greater than zero. There is one such defined by each line of route and they are called principal compositions. Their enumeration therefore coincides with that of the lines of route.

To each unipartite partition which has no part greater than p and not more than q parts corresponds a line of route. It has been shewn in Sect. IV, Ch. II, that the number of lines of route through the lattice is

$$\binom{p+q}{p}.$$

Hence the correspondence gives the theorem:

"The number of partitions, of all numbers, into parts limited in magnitude to p and in number to q is

$$\binom{p+q}{p}."$$

This number is also by Euler's intuitive theory given by the coefficient of $a^q x^{pq}$ or of $(ax^p)^q$ in the expansion of

$$\frac{1}{(1-a)(1-x)(1-ax)(1-ax^2)\dots(1-ax^p)}.$$

Hence that portion of this expansion which involves powers of ax^p is

$$1+\binom{p+1}{1}ax^p+\binom{p+2}{2}a^2x^{2p}+\dots+\binom{p+q}{p}a^qx^{pq}+\dots,$$

or $$(1-ax^p)^{-p-1}.$$

The number of different parts in the unipartite partition (denoted by crosses) is equal to the number of left-bends $\lrcorner$ in the line of route, and the number of lines of route which possess s left-bends (or s right-bends $\urcorner$) has been shewn in Sect. IV, Ch. II, Art. 145, to be

$$\binom{p}{s}\binom{q}{s}.$$

Hence the theorem is:

"The number of partitions of all numbers into s different parts limited in magnitude to p and in number to q is

$$\binom{p}{s}\binom{q}{s}."$$

This number is the coefficient of $\mu^s\alpha^p\beta^q$ in the product

$$(\alpha+\mu\beta)^p(\alpha+\beta)^q,$$

and therefore also by the Master Theorem of Sec. III, Ch. II in the expansion of

$$\frac{1}{1-\alpha-\beta+(1-\mu)\,\alpha\beta},$$

which may be written

$$\Sigma\frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}\mu^s.$$

Hence the function

$$\frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}$$

enumerates the lines of route, which possess s left-bends or s right-bends, in all bipartite reticulations. When the lattice is of the bipartite $\overline{pq}$ we merely seek the coefficient of $\alpha^p\beta^q$.

It also enumerates all unipartite partitions into s different parts limited in any desired manner in number and magnitude.

336. If we form the product

$$\left(1+\frac{bx}{1-x}\right)\left(1+\frac{bx^2}{1-x^2}\right)\dots\left(1+\frac{bx^p}{1-x^p}\right),$$

we find that the general term in the development is

$$bx^{\sigma_1}\,.\,bx^{2\sigma_2}\,.\,\dots\,bx^{p\sigma_p},$$

$\sigma_1, \sigma_2, \dots \sigma_p$ having any integer values from 0 to ∞. If s of these magnitudes be greater than zero this is equal to

$$b^s x^{\sigma_1+2\sigma_2+\dots+p\sigma_p},$$

and consequently the coefficient of $b^s x^n$ in the product denotes the number of ways of partitioning n into parts limited in magnitude to p and of exactly s *different* magnitudes, the number of parts being otherwise unrestricted.

If we modify the product in the manner

$$\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\dots\left(1+\frac{abx^p}{1-ax^p}\right)$$

and take the coefficient of $a^q b^s x^n$, the partitions possess the further property that the total number of parts is exactly q. If we add the additional factor $\frac{1}{1-a}$ the partitions enumerated by the coefficient of $a^q b^s x^n$ possess the property that the total number of parts is q or fewer.

Again if we also multiply the product by the factor $\frac{1}{1-x}$, the coefficient of $a^q b^s x^n$ enumerates the partitions of all numbers equal to or less than n, where the part magnitude is restricted so as not to exceed p, exactly s of the parts are different and the total number of parts is q or less. Hence the partitions of all unipartite numbers into exactly s different parts, limited in magnitude to p and in number to q, are enumerated by the coefficient of

$$a^q b^s x^{pq}$$

in the development of the product

$$\frac{1}{1-x}\cdot\frac{1}{1-a}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\dots\left(1+\frac{abx^p}{1-ax^p}\right).$$

This may be regarded as the intuitive solution of the problem after Euler.

This coefficient is equal to that of $(ax^p)^q$ in

$$\frac{a^s}{(1-x)(1-a)}\,\Sigma\,\frac{x^{k_1+k_2+\dots+k_s}}{(1-ax^{k_1})(1-ax^{k_2})\dots(1-ax^{k_s})},$$

where $k_1, k_2, \dots k_s$ are any s different numbers drawn from the series

$$1, 2, 3, \dots p,$$

and the summation is in respect of all such selections.

Inasmuch as we know from the reticulation theory that the coefficient is equal to

$$\binom{p}{s}\binom{q}{s},$$

we find that the effective portion of the above generating function is

$$\binom{p}{s}\binom{s}{s}(ax^p)^s+\binom{p}{s}\binom{s+1}{s}(ax^p)^{s+1}+\dots \text{ ad inf.}$$

and this is
$$\binom{p}{s}\frac{(ax^p)^s}{(1-ax^p)^{s+1}}.$$

Here we have regarded p and s as constant and q as variable; but if we take p and q to be constant and s variable we know that the coefficient of $(ax^p)^q$ in the product

$$\frac{1}{(1-x)(1-a)}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\dots\left(1+\frac{abx^p}{1-ax^p}\right)$$

is
$$\binom{p}{0}\binom{q}{0}+\binom{p}{1}\binom{q}{1}b+\dots+\binom{p}{q}\binom{q}{q}b^q.$$

A line of route with s left-bends has either $s-1$, s or $s+1$ right-bends. If it commences by tracing an α-segment and ends by tracing a β-segment the number is $s-1$. If it commences by tracing an α-segment and ends by tracing an α-segment *or* if it commences and ends by tracing β-segments the number is s. If a β-segment begins and an α-segment ends the line of route the number is $s+1$.

To obtain the correspondence with partitions we remark that if an α-segment begins the line, the partition (marked with crosses) has exactly q parts; if an α-segment ends, the highest part is less than p; if a β-segment begins, the number of parts is less than q; if a β-segment ends, the highest part is equal to p. It can moreover be readily shewn by the method pursued in Sect. IV, Ch. II, Art. 144, that the enumeration of the lines of route possessing s left-bends and $s-1$, s, $s+1$ right-bends respectively is given by the numbers

$$\binom{p-1}{s-1}\binom{q-1}{s-1},$$

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s},$$

$$\binom{p-1}{s}\binom{q-1}{s},$$

the sum of these numbers being of course $\binom{p}{s}\binom{q}{s}$.

337. Hence the following theorems:

"The number of partitions of all numbers which have exactly q parts, a highest part equal to p and s different parts, is

$$\binom{p-1}{s-1}\binom{q-1}{s-1}."$$

"The number of partitions of all numbers which have exactly q parts, a highest part less than p and s different parts, *or* which have less than q parts, a highest part equal to p and s different parts, is

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s}."$$

"The number of partitions of all numbers which have less than q parts, a highest part less than p, and s different parts, is

$$\binom{p-1}{s}\binom{q-1}{s}."$$

Ex. gr. If $p=q=3$, $s=2$, the partitions enumerated by these three theorems are

$$(311),\ (331),\ (322),\ (332);$$
$$(31),\ (211),\ (32),\ (221);$$
$$(21);$$

respectively.

It is clear that identical relations between binomial coefficients yield results in the theory of partitions.

Thus the relations

$$\binom{p+q}{p}-\binom{p+q-1}{p-1}=\binom{p+q-1}{p},$$
$$\binom{p-1}{s}\binom{q-1}{s}-\binom{p-2}{s}\binom{q-1}{s}=\binom{p-2}{s-1}\binom{q-1}{s},$$

admit of immediate interpretation.

By Euler's intuitive theory the enumerations in the three cases are given by

(i) the coefficient of $a^{q-1}b^{s-1}x^{pq-p}$ in

$$\frac{1}{1-x}\left\{1+\frac{abx}{1-ax}\right\}\left\{1+\frac{abx^2}{1-ax^2}\right\}\cdots\left\{1+\frac{abx^{p-1}}{1-ax^{p-1}}\right\}\left\{1+\frac{ax^p}{1-ax^p}\right\};$$

(ii) the coefficient of $a^q b^s x^{(p-1)q}$ in

$$\frac{1}{1-x}\left\{1+\frac{abx}{1-ax}\right\}\left\{1+\frac{abx^2}{1-ax^2}\right\}\cdots\left\{1+\frac{abx^{p-1}}{1-ax^{p-1}}\right\},$$

plus the coefficient of $a^{q-2}b^{s-1}x^{p(q-1)}$ in

$$\frac{1}{(1-x)(1-a)}\left\{1+\frac{abx}{1-ax}\right\}\left\{1+\frac{abx^2}{1-ax^2}\right\}\cdots\left\{1+\frac{abx^{p-1}}{1-ax^{p-1}}\right\}\left\{1+\frac{ax^p}{1-ax^p}\right\};$$

(iii) the coefficient of $a^{q-1}b^s x^{(p-1)(q-1)}$ in

$$\frac{1}{(1-x)(1-a)}\left\{1+\frac{abx}{1-ax}\right\}\left\{1+\frac{abx^2}{1-ax^2}\right\}\cdots\left\{1+\frac{abx^{p-1}}{1-ax^{p-1}}\right\}.$$

338. When the line of route is regarded as the zig-zag graph of a composition of the unipartite number $p+q+1$, it is convenient to rotate the figure clockwise through a right angle. The reading of the graph then shews $p+1$ parts, no part being greater than $q+1$.

ımber of the compositions so defined we know by the correspond-

$$\binom{p+q}{p}.$$

This is verified because the number is given by the

$$\text{Coefficient of } x^{p+q+1} \text{ in } (x + x^2 + \ldots + x^{q+1})^{p+1},$$
$$\text{or of } x^q \quad \text{in } (1-x)^{-p-1},$$

and this is
$$\binom{p+q}{p}.$$

In order from a composition of the unipartite to proceed to the unipartite partition, we write down the composition, say,

$$\begin{array}{cccccccc} 2 & 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 1 & & & & \\ & 1 & 2 & 3 & & & & \end{array} \quad \text{as in the diagram,}$$

and form a second row by subtracting unity from every part except the first, and then a third row of the natural numbers in order underneath the numbers in the second row; then the partition is formed by taking a number in the third row, repeated a number of times shewn by the number in the second row immediately above it; thus in the above case the partition is

$$3^1 \, 2^2 \, 1^2.$$

Generally if the composition be $\pi_1 \pi_2 \ldots \pi_{p+1}$ we form the scheme

$$\begin{array}{cccccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \ldots & \pi_{p-1} & \pi_p & \pi_{p+1} \\ & \pi_2 - 1 & \pi_3 - 1 & \pi_4 - 1 & \ldots & \pi_{p-1} - 1 & \pi_p - 1 & \pi_{p+1} - 1 \\ & 1 & 2 & 3 & \ldots & p-2 & p-1 & p \end{array}$$

and the corresponding partition is

$$p^{\pi_{p+1}-1} \; \overline{p-1}^{\pi_p - 1} \; \overline{p-2}^{\pi_{p-1}-1} \ldots 3^{\pi_4 - 1} \; 2^{\pi_3 - 1} \; 1^{\pi_2 - 1}.$$

Omitting the first part of the composition, if of the remainder there be s parts greater than unity, the partition will have s *different* parts. The total number of parts is $\Sigma\pi - \pi_1 - p$ or $q + 1 - \pi_1$.

SECTION VIII

A NEW BASIS OF THE THEORY OF PARTITIONS. PARTITION ANALYSIS IN SEVEN CHAPTERS

CHAPTER I

THE METHOD OF DIOPHANTINE INEQUALITIES

339. In this Section we no longer rest upon the intuitive observation of Euler, but penetrate deeper into the Theory of Partitions.

A partition of a number may be regarded as any collection of positive integers whose sum is equal to the number. There is no specification of order amongst the numbers which are the parts of the partition, and that being so, we may import into the definition any particular order or arrangement that is convenient. There are only two arrangements that are universally applicable. We may in all cases arrange the parts either in descending or in ascending order of numerical magnitude. We choose the former of these and make a new definition of a partition of a number, viz.:

"A partition of a number is any collection of positive integers *arranged in descending order of magnitude* whose sum is equal to the number."

The problem now is to find successions of numbers in descending order of magnitude where the numbers enjoy some other property which may be that their sum is to be equal to a given number. This is, in fact, the question immediately before us, but it is clearly not the only one that may present itself.

Consider i numbers in descending order of magnitude

$$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i.$$

The order is defined by the Diophantine relations

$$\begin{gathered} \alpha_1 \geqslant \alpha_2, \\ \alpha_2 \geqslant \alpha_3, \\ \vdots \\ \alpha_{i-1} \geqslant \alpha_i, \end{gathered}$$

and subject to them we consider the sum

$$\Sigma x^{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i}.$$

Now observe that the algebraic fraction

$$\frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\dots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x\right)},$$

when expanded in ascending powers of x, has the general term

$$(\lambda_1)^{\alpha_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2}\left(\frac{\lambda_3}{\lambda_2}\right)^{\alpha_3}\dots\left(\frac{\lambda_i}{\lambda_{i-1}}\right)^{\alpha_i} x^{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i},$$

or $$\lambda_1^{\alpha_1-\alpha_2}\lambda_2^{\alpha_2-\alpha_3}\dots\lambda_{i-1}^{\alpha_{i-1}-\alpha_i}\lambda_i^{\alpha_i}x^{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i},$$

and that if the Diophantine relations are to be satisfied this must be free from negative powers of $\lambda_1, \lambda_2, \lambda_3, \dots$.

It is thus evident that we have only to expand the fraction

$$\frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\dots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x\right)},$$

reject all the terms involving negative powers of $\lambda_1, \lambda_2, \lambda_3, \dots$, and subsequently put

$$\lambda_1=\lambda_2=\lambda_3=\dots=\lambda_i=1$$

to obtain the desired sum

$$\Sigma x^{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i}.$$

The performance of these operations upon the fraction we shall denote by prefixing the symbol $\underset{\geqslant}{\Omega}$, so that

$$\Sigma x^{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i}=\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\dots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x\right)}.$$

It will be noted that λ_i is retained although it may be immediately put equal to unity.

In regard to the operation of $\underset{\geqslant}{\Omega}$ we have the easily established result

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_1 A_1 x^{p_1})\left(1-\frac{A_2}{\lambda_1}x^{p_2}\right)}=\frac{1}{(1-A_1x^{p_1})(1-A_1A_2x^{p_1+p_2})},$$

leading to $$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_1 A_1 x^{p_1})\left(1-\frac{\lambda_2 A_2}{\lambda_1}x^{p_2}\right)\left(1-\frac{A_3}{\lambda_2}x^{p_3}\right)}$$

$$=\frac{1}{(1-A_1x^{p_1})(1-A_1A_2x^{p_1+p_2})(1-A_1A_2A_3x^{p_1+p_2+p_3})},$$

and to the similar general result involving any number of denominator factors.

In reaching this it will be observed that the auxiliaries λ are successively eliminated.

We thus find that

$$\Sigma x^{\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_i} = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^i)},$$

the generating function which with Euler was intuitive. The investigation above shews how the Theory of Partitions may be enlarged because we may vary the Diophantine relations, and also the linear function of the parts which above was taken to be the simple sum.

340. It has been shewn that we can step back from the generating function of Euler

$$\frac{1}{(1)(2)(3)\ldots(i)}$$

to the crude generating function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\ldots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x\right)},$$

and it will become evident that we can step back further still to a more crude generating function. This arises from the circumstance that

$$\overset{\mu}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\mu_1 x)\left(1-\frac{\mu_2}{\mu_1}\right)\left(1-\frac{\mu_3}{\mu_2}\right)\ldots\left(1-\frac{\mu_i}{\mu_{i-1}}\right)} = \frac{1}{(1-x)^i},$$

because
$$\overset{\mu}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\mu_1 x)\left(1-\frac{\mu_2}{\mu_1}\right)} = \frac{1}{(1-x)(1-\mu_2 x)},$$

$$\overset{\mu}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\mu_1 x)\left(1-\frac{\mu_2}{\mu_1}\right)\left(1-\frac{\mu_3}{\mu_2}\right)} = \frac{1}{(1-x)^2(1-\mu_3 x)},$$

and so on by successive elimination of $\mu_1, \mu_2, \ldots$.

With two sets of auxiliaries we have therefore for the sum

$$\overset{\lambda}{\underset{\geqslant}{\Omega}}\overset{\mu}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\lambda_1\mu_1 x)\left(1-\frac{\lambda_2\mu_2}{\lambda_1^2\mu_1}\right)\left(1-\frac{\lambda_1\lambda_3\mu_3}{\lambda_2^2\mu_2}\right)\left(1-\frac{\lambda_2\lambda_4\mu_4}{\lambda_3^2\mu_3}\right)\ldots\left(1-\frac{\lambda_{i-2}\lambda_i\mu_i}{\lambda_{i-1}^2\mu_{i-1}}\right)}.$$

Eliminating $\mu_1, \mu_2, \ldots \mu_i$ by the operation of $\overset{\mu}{\underset{\geqslant}{\Omega}}$ this becomes

$$\overset{\lambda}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\ldots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x\right)},$$

the function already considered.

To interpret the ultra-crude generating function above we form the general term

$$(\lambda_1\mu_1 x)^{\alpha_1}\left(\frac{\lambda_2\mu_2}{\lambda_1^2\mu_1}\right)^{\alpha_2}\left(\frac{\lambda_1\lambda_3\mu_3}{\lambda_2^2\mu_2}\right)^{\alpha_3}\left(\frac{\lambda_2\lambda_4\mu_4}{\lambda_3^2\mu_3}\right)^{\alpha_4}\cdots\left(\frac{\lambda_{i-2}\lambda_i\mu_i}{\lambda_{i-1}^2\mu_{i-1}}\right)^{\alpha_i},$$

which is

$$\lambda_1^{\alpha_1-2\alpha_2+\alpha_3}\lambda_2^{\alpha_2-2\alpha_3+\alpha_4}\lambda_3^{\alpha_3-2\alpha_4+\alpha_5}\cdots\lambda_{i-2}^{\alpha_{i-2}-2\alpha_{i-1}+\alpha_i}\lambda_{i-1}^{\alpha_{i-1}-2\alpha_i}\lambda_i^{\alpha_i}\mu_1^{\alpha_1-\alpha_2}\mu_2^{\alpha_2-\alpha_3}\cdots$$

$$\cdots\mu_{i-1}^{\alpha_{i-1}-\alpha_i}\mu_i^{\alpha_i}x^{\alpha_1},$$

and since the exponents must be non-negative we have the Diophantine relations

$$\left.\begin{aligned}\alpha_1+\alpha_3&\geqslant 2\alpha_2\\ \alpha_2+\alpha_4&\geqslant 2\alpha_3\\ &\vdots\\ \alpha_{i-2}+\alpha_i&\geqslant 2\alpha_{i-1}\\ \alpha_{i-1}&\geqslant 2\alpha_i\end{aligned}\right\},$$

$$\alpha_1\geqslant\alpha_2\geqslant\alpha_3\geqslant\ldots\geqslant\alpha_i,$$

in connexion with the sum Σx^{α_1}.

If then $\alpha_1, \alpha_2, \ldots \alpha_i$ be any i integers, which satisfy the above set of relations, Σx^{α_1}, the sum being in respect of each set of integers, is equal to

$$\frac{1}{(1)\,(2)\,(3)\ldots(i)}.$$

This shews that the number of ways of choosing the integers, the number α_1 being fixed, is equal to the number of partitions of α_1 into i or fewer parts.

Ex. gr. take $\alpha_1 = 4$, $i = 4$, the sets of integers are

α_1	α_2	α_3	α_4
4	0	0	0
4	1	0	0
4	2	0	0
4	2	1	0
4	3	2	1

in each of which the Diophantine relations are satisfied. They are five in number, because the number four has five partitions into four or fewer parts. In fact, to make the one-to-one correspondence between the sets and the partitions of four we have merely to take the numbers

$$\alpha_1-\alpha_2,\ \alpha_2-\alpha_3,\ \alpha_3-\alpha_4,\ \alpha_4.$$

In general we may take a set of integers enumerated by the ultra-crude form to be

$$\beta_1+\beta_2+\ldots+\beta_i,\ \beta_2+\beta_3+\ldots+\beta_i,\ \beta_3+\ldots+\beta_i,\ldots\beta_{i-1}+\beta_i,\ \beta_i,$$

which plainly satisfies the Diophantine relations, and then the corresponding partition of α_1 is

$$\beta_1, \beta_2, \beta_3, \ldots \beta_i.$$

341. As another example the generating function

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots(1-x^{2i-1})}$$

of the partitions composed of uneven parts not exceeding $2i-1$ in magnitude, may be exhibited in the crude form

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x^2\right)\left(1-\frac{\lambda_3}{\lambda_2}x^2\right)\ldots\left(1-\frac{\lambda_i}{\lambda_{i-1}}x^2\right)},$$

which is equal to the sum

$$\Sigma x^{\alpha_1+2\alpha_2+2\alpha_3+\ldots+2\alpha_i},$$

subject to the Diophantine conditions

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_{i-1} \geqslant \alpha_i.$$

Ex. gr. for $i=3$, $\alpha_1+2\alpha_2+2\alpha_3=6$, it enumerates the partitions

$$22,\ 211,\ 41,\ 6,$$

equi-numerous with the partitions of 6 into uneven parts.

We further proceed to the ultra-crude form

$$\overset{\lambda}{\underset{\geqslant}{\Omega}}\ \overset{\mu}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\lambda_1\mu_1 x)\left(1-\frac{\lambda_2\mu_2}{\lambda_1^2\mu_1}x\right)\left(1-\frac{\lambda_1\lambda_3\mu_3}{\lambda_2^2\mu_2}\right)\left(1-\frac{\lambda_2\lambda_4\mu_4}{\lambda_3^2\mu_3}\right)\ldots\left(1-\frac{\lambda_{i-2}}{\lambda_{i-1}^2\mu_{i-1}}\right)},$$

since the elimination of the auxiliaries μ leads to the crude form.

This is the sum $\qquad \Sigma x^{\alpha_1+\alpha_2},$

for sets of integers $\qquad \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i,$

which are subject to the Diophantine relations

$$\left.\begin{aligned} \alpha_1+\alpha_3 &\geqslant 2\alpha_2 \\ \alpha_2+\alpha_4 &\geqslant 2\alpha_3 \\ &\vdots \\ \alpha_{i-2}+\alpha_i &\geqslant 2\alpha_{i-1} \\ \alpha_{i-1} &\geqslant 2\alpha_i \end{aligned}\right\}$$

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_{i-1} \geqslant \alpha_i.$$

The number of such sets of integers for which $\alpha_1+\alpha_2$ has a given value is equal to the number of partitions of $\alpha_1+\alpha_2$ into uneven parts, no part exceeding $2i-1$ in magnitude.

Similarly the function

$$\frac{1}{(1-x)(1-x^6)(1-x^{11})\dots(1-x^{5i-4})}$$

has the crude form

$$\underset{\geqslant}{\overset{\lambda}{\Omega}} \frac{1}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x^5\right)\left(1-\frac{\lambda_3}{\lambda_2}x^5\right)\dots\left(1-\frac{1}{\lambda_{i-1}}x^5\right)},$$

which denotes the sum

$$\Sigma x^{\alpha_1+5\alpha_2+5\alpha_3+\dots+5\alpha_i},$$

subject to the conditions

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \dots \geqslant \alpha_i,$$

and the ultra-crude form

$$\underset{\geqslant}{\overset{\lambda}{\Omega}}\underset{\geqslant}{\overset{\mu}{\Omega}} \frac{1}{(1-\lambda_1\mu_1 x)\left(1-\frac{\lambda_2\mu_2}{\lambda_1^2\mu_1}x^4\right)\left(1-\frac{\lambda_1\lambda_3\mu_3}{\lambda_2^2\mu_2}\right)\left(1-\frac{\lambda_2\lambda_4\mu_4}{\lambda_3^2\mu_3}\right)\dots\left(1-\frac{\lambda_{i-2}}{\lambda_{i-1}^2\mu_{i-1}}\right)},$$

which denotes the sum $\Sigma x^{\alpha_1+4\alpha_2}$

for sets of integers $\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots\ \alpha_i,$

which obey the conditions

$$\begin{gathered}\alpha_1+\alpha_3 \geqslant 2\alpha_2,\\ \alpha_2+\alpha_4 \geqslant 2\alpha_3,\\ \vdots\\ \alpha_{i-2}+\alpha_i \geqslant 2\alpha_{i-1},\\ \alpha_{i-1} \geqslant 2\alpha_i,\\ \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \dots \geqslant \alpha_i.\end{gathered}$$

The number of such sets of integers for which $\alpha_1+4\alpha_2$ has a given value is equal to the number of partitions of $\alpha_1+4\alpha_2$ into parts of the form $5m+1$, no part exceeding $5i-4$ in magnitude.

In general we find the crude form of the sum

$$\Sigma x^{\theta_1\alpha_1+\theta_2\alpha_2+\dots+\theta_i\alpha_i},$$

where $\theta_1, \theta_2, \dots \theta_i$ are positive or negative integers, subject to the Diophantine relation

$$\sigma_1\alpha_1+\sigma_2\alpha_2+\dots+\sigma_i\alpha_i \geqslant 0,$$

where $\sigma_1, \sigma_2, \dots \sigma_i$ are positive or negative integers, to be

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1^{\sigma_1}x^{\theta_1})(1-\lambda_1^{\sigma_2}x^{\theta_2})\dots(1-\lambda_1^{\sigma_i}x^{\theta_i})}.$$

If we have an additional Diophantine condition

$$\tau_1\alpha_1+\tau_2\alpha_2+\dots+\tau_i\alpha_i \geqslant 0,$$

we intercalate another auxiliary λ_2, making the function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1^{\sigma_1}\lambda_2^{\tau_1}x^{\theta_1})(1-\lambda_1^{\sigma_2}\lambda_2^{\tau_2}x^{\theta_2})\dots(1-\lambda_1^{\sigma_i}\lambda_2^{\tau_i}x^{\theta_i})},$$

and so forth, a new auxiliary appearing for each additional Diophantine relation.

342. The generating functions which have been considered above enumerate the partitions specified. They are enumerating generating functions. We can form real generating functions which put in evidence the partitions enumerated in the following manner.

The general term in the expansion of the fraction

$$\frac{1}{(1-\lambda_1 X_1)\left(1-\frac{\lambda_2}{\lambda_1}X_2\right)\left(1-\frac{\lambda_3}{\lambda_2}X_3\right)\dots\left(1-\frac{\lambda_{i-1}}{\lambda_{i-2}}X_{i-1}\right)\left(1-\frac{1}{\lambda_{i-1}}X_i\right)}$$

is
$$(\lambda_1 X_1)^{\alpha_1}\left(\frac{\lambda_2}{\lambda_1}X_2\right)^{\alpha_2}\left(\frac{\lambda_3}{\lambda_2}X_3\right)^{\alpha_3}\dots\left(\frac{\lambda_{i-1}}{\lambda_{i-2}}X_{i-1}\right)^{\alpha_{i-1}}\left(\frac{1}{\lambda_{i-1}}X_i\right)^{\alpha_i},$$

or
$$\lambda_1^{\alpha_1-\alpha_2}\lambda_2^{\alpha_2-\alpha_3}\dots\lambda_{i-1}^{\alpha_{i-1}-\alpha_i}X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_{i-1}^{\alpha_{i-1}}X_i^{\alpha_i}.$$

If the numbers $\alpha_1, \alpha_2, \dots \alpha_i$ are subject to the Diophantine relations

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \dots \geqslant \alpha_{i-1} \geqslant \alpha_i,$$

it is clear that the product

$$X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_i^{\alpha_i}$$

indicates a partition $(\alpha_1\alpha_2\alpha_3\dots\alpha_i)$ of the number $\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_i$, and

$$\Sigma X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_i^{\alpha_i}$$

can be represented in the crude form

$$\overset{\lambda}{\underset{\geqslant}{\Omega}} \frac{1}{(1-\lambda_1 X_1)\left(1-\frac{\lambda_2}{\lambda_1}X_2\right)\left(1-\frac{\lambda_3}{\lambda_2}X_3\right)\dots\left(1-\frac{\lambda_{i-1}}{\lambda_{i-2}}X_{i-1}\right)\left(1-\frac{1}{\lambda_{i-1}}X_i\right)},$$

which by a previous Article has the expression

$$\frac{1}{(1-X_1)(1-X_1X_2)(1-X_1X_2X_3)\dots(1-X_1X_2X_3\dots X_i)}.$$

We observe herein that

$$X_1^{\alpha_1-\alpha_2}(X_1X_2)^{\alpha_2-\alpha_3}(X_1X_2X_3)^{\alpha_3-\alpha_4}\dots(X_1X_2\dots X_{i-1})^{\alpha_{i-1}-\alpha_i}(X_1X_2\dots X_i)^{\alpha_i}$$
$$= X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_i^{\alpha_i},$$

and that the denominator factors establish that the partitions under consideration are uniquely composable by means of the partitions

1, 11, 111, ..., 111...i units.

Ex. gr. To obtain the partition

$$\alpha_1\alpha_2\alpha_3 \dots \alpha_i,$$

we take 1, $\alpha_1 - \alpha_2$ times; 11, $\alpha_2 - \alpha_3$ times, and so on.

We have thus the algebraic indication of the Ferrers' graphical representation of a partition in which the nodes are replaced by units, and we further learn that the partitions 1, 11, 111, ... are the fundamental or ground partitions from which all others of the nature considered may be derived by addition.

These remarks may appear, in the present instance, to be of a trivial character, but they involve an *idea* which will be found to be of great importance as the theory advances.

We can also proceed to an ultra-crude form

$$\underset{\geqslant}{\overset{\lambda}{\Omega}}\;\underset{\geqslant}{\overset{\mu}{\Omega}}\;\frac{1}{(1-\lambda_1\mu_1 X_1)\left(1-\dfrac{\lambda_2\mu_2}{\lambda_1^2\mu_1}\dfrac{X_2}{X_1}\right)\left(1-\dfrac{\lambda_1\lambda_3\mu_3}{\lambda_2^2\mu_2}\dfrac{X_3}{X_2}\right)\dots},$$

the first of an infinite series of ultra-crude forms. The next one is

$$\underset{\geqslant}{\overset{\lambda}{\Omega}}\;\underset{\geqslant}{\overset{\mu}{\Omega}}\;\underset{\geqslant}{\overset{\nu}{\Omega}}\;\frac{1}{(1-\lambda_1\mu_1\nu_1 X_1)\left(1-\dfrac{\lambda_2\mu_2\nu_2}{\lambda_1^3\mu_1^2\nu_1}\dfrac{X_2}{X_1^2}\right)\left(1-\dfrac{\lambda_1^3\mu_1\lambda_3\mu_3\nu_3}{\lambda_2^3\mu_2^2\nu_2}\dfrac{X_1X_3}{X_2^2}\right)\dots},$$

and succeeding ones are written down without difficulty.

Each one is interpretable in terms of Diophantine inequalities by constructing the general term.

343. We pass on to the consideration of the partition of numbers into parts limited not to exceed j in magnitude.

It is merely necessary to restrict the highest part and we are led to the crude enumerating function

$$\underset{\geqslant}{\Omega}\;\frac{1-(\lambda_1 x)^{j+1}}{(1-\lambda_1 x)\left(1-\dfrac{\lambda_2}{\lambda_1}x\right)\dots\left(1-\dfrac{x}{\lambda_{i-1}}\right)};$$

the partitions possessing at most i parts.

Instead of at once evaluating this expression we notice the identity

$$\frac{1}{(1-g)(1-g\lambda_1 x)} = \sum_{j=0}^{j=\infty} g^j\,\frac{1-(\lambda_1 x)^{j+1}}{1-\lambda_1 x},$$

and observe that the crude function is equal to the coefficient of g^j in

$$\underset{\geqslant}{\Omega}\;\frac{1}{(1-g)(1-g\lambda_1 x)\left(1-\dfrac{\lambda_2}{\lambda_1}x\right)\left(1-\dfrac{\lambda_3}{\lambda_2}x\right)\dots\left(1-\dfrac{x}{\lambda_{i-1}}\right)},$$

or in
$$\frac{1}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^i)},$$

the well-known intuitive result of Euler.

We apply the same method to reach results which are generalizations of the foregoing.

Let us restrict the magnitudes of the successive parts, written in descending order, by the numbers

$$j_1, j_2, \dots j_i.$$

We are led to the crude enumerating function

$$\underset{\geqslant}{\Omega} \frac{\{1-(\lambda_1 x)^{j_1+1}\}\left\{1-\left(\frac{\lambda_2}{\lambda_1}x\right)^{j_2+1}\right\}\dots\left\{1-\frac{x}{\lambda_{i-1}}\right\}^{j_i+1}}{(1-\lambda_1 x)\quad\left(1-\frac{\lambda_2}{\lambda_1}x\right)\quad\dots\quad\left(1-\frac{x}{\lambda_{i-1}}\right)},$$

which we notice is the coefficient of

$$g_1^{j_1} g_2^{j_2} \dots g_i^{j_i}$$

in the function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-g_1)(1-g_2)\dots(1-g_i)(1-g_1\lambda_1 x)\left(1-g_2\frac{\lambda_2}{\lambda_1}x\right)\dots\left(1-g_i\frac{x}{\lambda_{i-1}}\right)},$$

and this function is

$$\frac{1}{(1-g_1)(1-g_2)\dots(1-g_i)(1-g_1x)(1-g_1g_2x^2)(1-g_1g_2g_3x^3)\dots(1-g_1g_2\dots g_ix^i)}.$$

We have therefore to expand this function and take the coefficient of $g_1^{j_1} g_2^{j_2} \dots g_i^{j_i}$. There are two remarks to be made here.

344. Firstly the function lends itself to intuitive interpretation after Euler. For clearly the factor $g_s^{j_s}$ of $g_1^{j_1} g_2^{j_2} \dots g_i^{j_i}$ points out that parts not less than s in magnitude occur in the partitions j_s or fewer times. This is seen at once by merely looking at the denominator factors which involve g_s. Hence by the intuitive interpretation the partitions enumerated are those in which the part magnitude in the whole partition is restricted not to exceed i and parts not less than s in magnitude occur j_s or fewer times, where s has any of the values 1, 2, 3, ... i.

The same function also enumerates, by this investigation, partitions into i or fewer parts, any part s occurring at most j_s times, where s has any of the values 1, 2, 3, ... i.

We have thus a theorem of reciprocity the generalization of that in regard to partitions restricted as to part magnitude and as to number of parts.

The reader will notice that this generalized reciprocity is made evident by the graph of a partition. It simply depends upon the circumstance that the graph may be read either by rows or by columns.

Secondly the function, in the form to which it has been brought, is still crude because the numbers $j_1, j_2, \dots j_i$ are necessarily in descending order and we require therefore only those products of powers of $g_1, g_2, \dots g_i$ which arise in the expansion for which this descending order is in evidence.

Taking a new set of auxiliaries $\nu_1, \nu_2, \dots \nu_{i-1}$ the true generating function may be written

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\nu_1 g_1)\left(1-\frac{\nu_2}{\nu_1}g_2\right)\dots\left(1-\frac{g_i}{\nu_{i-1}}\right)(1-\nu_1 g_1 x)(1-\nu_2 g_1 g_2 x^2)\dots(1-g_1 g_2\dots g_i x^i)}.$$

Eliminating ν_1 we find

$$\underset{\geqslant}{\Omega}\frac{1-\nu_2 g_1^2 g_2 x}{(1-g_1)(1-\nu_2 g_1 g_2)\left(1-\frac{\nu_3}{\nu_2}g_3\right)\dots\left(1-\frac{g_i}{\nu_{i-1}}\right)(1-g_1 x)(1-\nu_2 g_1 g_2 x)(1-\nu_2 g_1 g_2 x^2)\dots(1-g_1 g_2\dots g_i}$$

and we can continue to eliminate $\nu_2, \nu_3, \dots$ in succession. By doing this, however, we rapidly arrive at very complicated expressions. We can, however, eliminate ν_3 now without difficulty. The function becomes

$$\underset{\geqslant}{\Omega}\frac{(1-\nu_2 g_1^2 g_2 x)\left(1-\frac{\nu_4}{\nu_2}g_1 g_2 g_3^2 g_4 x^3\right)}{(1-g_1)(1-\nu_2 g_1 g_2)\left(1-\frac{g_3}{\nu_2}\right)\left(1-\frac{\nu_4}{\nu_2}g_3 g_4\right)\left(1-\frac{\nu_5}{\nu_4}g_5\right)\dots\left(1-\frac{g_i}{\nu_{i-1}}\right)}$$
$$\times(1-g_1 x)(1-\nu_2 g_1 g_2 x)(1-\nu_2 g_1 g_2 x^2)(1-g_1 g_2 g_3 x^3)$$
$$(1-\nu_4 g_1 g_2 g_3 g_4 x^3)(1-\nu_4 g_1 g_2 g_3 g_4 x^4)\dots,$$

and $\nu_5, \nu_7, \dots$ can be eliminated without trouble.

The even suffixed auxiliaries however introduce complexity.

345. The case in which the two highest parts are the only ones restricted is worth a moment's attention.

The true generating function is then

$$\frac{1-g_1^2 g_2 x}{(1-g_1)(1-g_1 x)(1-g_1 g_2)(1-g_1 g_2 x)(1-g_1 g_2 x^2)\dots(1-g_1 g_2 x^i)}.$$

The coefficient of $g_1^{j_1} g_2^{j_2}$ in the part

$$\frac{1}{(1-g_1 g_2)(1-g_1 g_2 x)\dots(1-g_1 g_2 x^i)},$$

is
$$\frac{(i+1)(i+2)\dots(i+j_2)}{(1)(2)\dots(j_2)},$$

and thence the coefficient of the product $g_1^{j_1} g_2^{j_2}$ in the whole fraction is

$$\frac{(i+1)(i+2)\dots(i+j_2)(j_1-j_2+1)}{(1)^2(2)\dots(j_2)}-x\frac{(i+1)(i+2)\dots(i+j_2-1)(j_1-j_2)}{(1)^2(2)\dots(j_2-1)},$$

or
$$\frac{(i+1)(i+2)\dots(i+j_2-1)}{(1)^2(2)\dots(j_2)}\{(i+j_2)(j_1-j_2+1)-x(j_2)(j_1-j_2)\},$$

a function which reduces when $j_2 = j_1 = j$ to the well-known enumerator

$$\frac{(i+1)(i+2)\dots(i+j)}{(1)(2)\dots(j)}.$$

346. The function which enumerates the number of ways of partitioning a number into j or fewer uneven parts, no part being greater than $2i-1$, is by intuition

$$\frac{1}{(1-g)(1-gx)(1-gx^3)\dots(1-gx^{2i-1})},$$

in which we have to seek the coefficient of g^j.

This function is

$$\underset{\geqslant}{\Omega}\frac{1}{(1-g)(1-g\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x^2\right)\left(1-\frac{\lambda_3}{\lambda_2}x^2\right)\dots\left(1-\frac{x^2}{\lambda_{i-1}}\right)},$$

and herein the coefficient of g^j is

$$\underset{\geqslant}{\Omega}\frac{1-(\lambda_1 x)^{j+1}}{(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x^2\right)\left(1-\frac{\lambda_3}{\lambda_2}x^2\right)\dots\left(1-\frac{x^2}{\lambda_{i-1}}\right)}.$$

This denotes the sum

$$\Sigma x^{\alpha_1+2\alpha_2+2\alpha_3+\dots+2\alpha_i},$$

subject to the Diophantine relations

$$j \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_i.$$

This summation, in consequence, depends upon partitions into uneven parts limited both in number and magnitude.

The Ω operations.

347. The operations that are of chief use in connexion with the Diophantine equalities and inequalities which present themselves in many of the ensuing chapters of this book are $\underset{\geqslant}{\Omega}, \underset{=}{\Omega}, \underset{>}{\Omega}$; but to these we may add as occasion arises $\underset{\leqslant}{\Omega}, \underset{<}{\Omega}, \underset{\gtrless}{\Omega}$.

The operation $\underset{\geqslant}{\Omega}$ has been already defined as cancelling all terms of expansions which involve negative powers of certain auxiliary quantities and in the remaining terms, which involve positive or zero powers of such quantities, putting each of the quantities equal to unity.

The operation $\underset{=}{\Omega}$ retains only terms which involve zero powers of such quantities.

The operation $\underset{>}{\Omega}$ retains only positive powers and then puts each quantity equal to unity and so forth.

A short study of these operations is essential.

The operation $\underset{\geqslant}{\Omega}$.

In connexion with the inequality

$$\alpha_1 \geqslant \alpha_2,$$

we have already found that the sum

$$\Sigma x^{\alpha_1} y^{\alpha_2}$$

depends upon $$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

348. We may add conveniently the easily verifiable results

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)(1-xz)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^2}\right)} = \frac{1}{(1-x)(1-x^2y)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy}{(1-x)(1-xy^2)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda^2}\right)} = \frac{1+xyz-x^2yz-xy^2z}{(1-x)(1-y)(1-x^2z)(1-y^2z)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)} = \frac{1}{(1-x)(1-x^sy)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy\dfrac{1-y^{s-1}}{1-y}}{(1-x)(1-xy^s)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1+xy+xz+xyz}{(1-x)(1-xy^2)(1-xz^2)},$$

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^2 x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)} = \frac{1+xz-xyz-xyz^2}{(1-x)(1-y)(1-yz)(1-xz^2)},$$

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-\lambda z)\left(1-\frac{w}{\lambda}\right)}$$

$$=\frac{1-xyw-xzw-yzw+xyzw+xyzw^2}{(1-x)(1-y)(1-z)(1-xw)(1-yw)(1-zw)},$$

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)\left(1-\frac{w}{\lambda}\right)}$$

$$=\frac{1-xyz-xyw-xyzw+xy^2zw+x^2yzw}{(1-x)(1-y)(1-xz)(1-xw)(1-yz)(1-yw)}.$$

349. A very useful principle is that of adding an inequality which is *à fortiori* true. Thus considering the inequality

$$\alpha_1 \geqslant 2\alpha_2,$$

which leads to
$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^2}\right)},$$

we may add the inequality
$$\alpha_1 \geqslant \alpha_2,$$

which is *à fortiori* true if the former be true, and thence we are led to

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda\mu x)\left(1-\frac{y}{\lambda^2\mu}\right)}.$$

Eliminating μ this is

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{xy}{\lambda}\right)},$$

and now eliminating λ this is

$$\frac{1}{(1-x)(1-x^2y)},$$

obtained by two applications of the simplest theorem of the subject.

Similarly any number of inequalities which are *à fortiori* true may be added with the object of reducing the problem to one of greater simplicity. Other useful principles present themselves as the subject advances.

350. Every Diophantine inequality is expressible as a Diophantine equality. For it is clear that

$$A\alpha_1 + B\alpha_2 \geqslant C\alpha_3$$

is equivalent to
$$A\alpha_1 + B\alpha_2 = C\alpha_3 + \alpha_4,$$

it being postulated that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are positive integers or zero.

Moreover every Diophantine equality is expressible in the form of two simultaneous Diophantine inequalities. For

$$A\alpha_1 + B\alpha_2 = C\alpha_3 + D\alpha_4$$

is clearly equivalent to the pair of inequalities

$$A\alpha_1 + B\alpha_2 \geqslant C\alpha_3 + D\alpha_4,$$

$$C\alpha_3 + D\alpha_4 \geqslant A\alpha_1 + B\alpha_2.$$

Thus $\Sigma x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ for the inequality

$$A\alpha_1 + B\alpha_2 \geqslant C\alpha_3$$

is
$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^A x_1)(1-\lambda^B x_2)\left(1-\frac{x_3}{\lambda^C}\right)},$$

and $\Sigma x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$ for the equality

$$A\alpha_1 + B\alpha_2 = C\alpha_3 + \alpha_4$$

is
$$\underset{=}{\Omega} \frac{1}{(1-\lambda^A x_1)(1-\lambda^B x_2)\left(1-\frac{x_3}{\lambda^C}\right)\left(1-\frac{x_4}{\lambda}\right)},$$

which is the same as the former with the additional factor in each term

$$x_4^{A\alpha_1+B\alpha_2-C\alpha_3}.$$

Also
$$\underset{=}{\Omega} \frac{1}{(1-\lambda^A x_1)(1-\lambda^B x_2)\left(1-\frac{x_3}{\lambda^C}\right)\left(1-\frac{x_4}{\lambda^D}\right)}$$

$$\equiv \underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda^A}{\mu^A}x_1\right)\left(1-\frac{\lambda^B}{\mu^B}x_2\right)\left(1-\frac{\mu^C}{\lambda^C}x_3\right)\left(1-\frac{\mu^D}{\lambda^D}x_4\right)}.$$

351. In general, if the operand be $F(\lambda)$, it is easily seen that

$$\underset{\geqslant}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} F\left(\frac{1}{\lambda}\right),$$

$$\underset{\leqslant}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} F\left(\frac{1}{\lambda}\right),$$

$$F(1) = \underset{\geqslant}{\Omega} F(\lambda) + \underset{\leqslant}{\Omega} F(\lambda) - \underset{=}{\Omega} F(\lambda),$$

and thence
$$\underset{=}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} F(\lambda) + \underset{\geqslant}{\Omega} F\left(\frac{1}{\lambda}\right) - F(1),$$

a result of much service.

Also
$$\underset{=}{\Omega} F(\lambda) = \underset{=}{\Omega} F\left(\frac{1}{\lambda}\right),$$

$$\underset{>}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} F(\lambda) - \underset{=}{\Omega} F(\lambda) = F(1) - \underset{\geqslant}{\Omega} F\left(\frac{1}{\lambda}\right),$$

$$\underset{>}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} \frac{1}{\lambda} F(\lambda).$$

Making use of these theorems we obtain the formulæ

$$\underset{=}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)}=\frac{1}{1-xy},$$

$$\underset{=}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)}=\frac{1}{(1-xy)(1-xz)},$$

$$\underset{=}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)}=\frac{1}{1-x^sy},$$

$$\underset{=}{\Omega}\frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)}=\frac{1+xyz}{(1-xy^2)(1-xz^2)},$$

$$\underset{=}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-\lambda z)\left(1-\frac{w}{\lambda}\right)}=\frac{1}{(1-xw)(1-yw)(1-zw)},$$

$$\underset{=}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)\left(1-\frac{w}{\lambda}\right)}=\frac{1-xyzw}{(1-xz)(1-xw)(1-yz)(1-yw)}.$$

352. Also in connexion with $\alpha_1+\alpha_2>\alpha_3$

$$\Sigma x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}=\underset{>}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)}$$

$$=\underset{\geqslant}{\Omega}\frac{\frac{1}{\lambda}}{(1-\lambda x)(1-\lambda y)\left(1-\frac{z}{\lambda}\right)}$$

$$=\frac{x+y-xy-xyz}{(1-x)(1-y)(1-xz)(1-yz)}.$$

353. $$\underset{>}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)}=\frac{x}{(1-x)(1-xy)(1-xz)},$$

$$\underset{>}{\Omega}\frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)}=\frac{x}{1-x^sy},$$

$$\underset{>}{\Omega}\frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)}=\frac{x(1-y^s)}{(1-x)(1-y)(1-xy^s)},$$

$$\underset{>}{\Omega}\frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)}=\frac{x(1+y+z+xyz)}{(1-x)(1-xy^2)(1-xz^2)}.$$

354. To these results may be added

$$\underset{\geqslant}{\Omega} \frac{\dfrac{1}{\lambda^s}}{(1-\lambda x)\left(1-\dfrac{y}{\lambda}\right)} = \frac{x^s}{(1-x)(1-xy)},$$

$$\underset{\geqslant}{\Omega} \frac{\lambda^s}{(1-\lambda x)\left(1-\dfrac{y}{\lambda}\right)} = \frac{1-xy-y^{s+1}+xy^{s+1}}{(1-x)(1-y)(1-xy)},$$

corresponding to

$$\alpha_1 \geqslant \alpha_2 + s,$$
$$\alpha_1 + s \geqslant \alpha_2,$$

respectively.

CHAPTER II

A SYZYGETIC THEORY

355. The simple theory of unipartite partitions has been made to depend upon s Diophantine inequalities

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_s \geqslant 0,$$

s being an arbitrary integer.

We enlarge the theory by making the integers $\alpha_1, \alpha_2, \alpha_3, \ldots$ depend upon a number of inequalities

$$\begin{aligned}
&A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + \ldots + A_s^{(1)}\alpha_s \geqslant 0,\\
&A_1^{(2)}\alpha_1 + A_2^{(2)}\alpha_2 + \ldots + A_s^{(2)}\alpha_s \geqslant 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&A_1^{(r)}\alpha_1 + A_2^{(r)}\alpha_2 + \ldots + A_s^{(r)}\alpha_s \geqslant 0,
\end{aligned}$$

which involve at most rs numerical magnitudes A, each of which may be positive, zero, or negative; but in each inequality it is clear that one at least must be positive.

For all sets of numbers $\alpha_1, \alpha_2, \ldots \alpha_s$ which satisfy the inequalities we seek the sum

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_s^{\alpha_s}.$$

By a theorem of Hilbert it appears that there is in every case a finite number of ground or fundamental solutions of the inequalities, viz.:

$$\begin{matrix}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \ldots \alpha_s^{(1)},\\
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \ldots \alpha_s^{(2)},\\
\ldots & \ldots & \ldots & \ldots\\
\alpha_1^{(m)} & \alpha_2^{(m)} & \alpha_3^{(m)} & \ldots \alpha_s^{(m)},
\end{matrix}$$

such that every solution $\qquad \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_s$

is of the form

$$\begin{aligned}
\alpha_1 &= c_1\alpha_1^{(1)} + c_2\alpha_1^{(2)} + \ldots + c_m\alpha_1^{(m)},\\
\alpha_2 &= c_1\alpha_2^{(1)} + c_2\alpha_2^{(2)} + \ldots + c_m\alpha_2^{(m)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\alpha_s &= c_1\alpha_s^{(1)} + c_2\alpha_s^{(2)} + \ldots + c_m\alpha_s^{(m)},
\end{aligned}$$

$c_1, c_2, c_3, \ldots c_m$ being positive integers.

This arises from the circumstance that every term

$$X_1^{a_1} X_2^{a_2} X_3^{a_3} \dots X_s^{a_s}$$

of the summation is found to be expressible as a product

$$\begin{aligned} &\{X_1^{a_1^{(1)}} X_2^{a_2^{(1)}} X_3^{a_3^{(1)}} \dots X_s^{a_s^{(1)}}\}^{c_1} \\ \times &\{X_1^{a_1^{(2)}} X_2^{a_2^{(2)}} X_3^{a_3^{(2)}} \dots X_s^{a_s^{(2)}}\}^{c_2} \\ \times &\dots\dots\dots\dots\dots\dots \\ \times &\{X_1^{a_1^{(m)}} X_2^{a_2^{(m)}} X_3^{a_3^{(m)}} \dots X_s^{a_s^{(m)}}\}^{c_m}. \end{aligned}$$

Denoting this product by

$$P_1^{c_1} P_2^{c_2} P_3^{c_3} \dots P_m^{c_m},$$

the sum (or generating function of solutions) takes the form

$$\frac{1 - \{Q_1^{(1)} + Q_1^{(2)} + Q_1^{(3)} + \dots\} + \{Q_2^{(1)} + Q_2^{(2)} + Q_2^{(3)} + \dots\} - \{Q_3^{(1)} + Q_3^{(2)} + Q_3^{(3)} + \dots\} + \dots}{(1 - P_1)(1 - P_2)(1 - P_3) \dots (1 - P_m)},$$

and we have what is termed a *syzygetic theory.*

When a number of algebraic expressions which involve the same numerical magnitudes are not linearly independent, but are connected by a linear relation, they are said (borrowing a word from Astronomy) to be in *syzygy.* In the above sum the expressions

$$P_1, P_2, P_3, \dots P_m$$

are all different, but it might happen that

$$P_1P_3 - P_2P_4 = 0.$$

This would be termed a syzygy, and clearly the sum we are seeking cannot have the form

$$\frac{1}{(1 - P_1)(1 - P_2)(1 - P_3) \dots (1 - P_m)},$$

for this would sum the term (or expression) P_1P_3 twice over. The syzygy before us shews that $1 - P_1P_3$ must be a portion of the numerator of the sum, so that the term P_1P_3 may be only accounted for once in the sum. The syzygy is called further a *first syzygy.* There may be more than one, and they account for the numerator terms

$$- Q_1^{(1)} - Q_1^{(2)} - Q_1^{(3)} - \dots.$$

Moreover, these first syzygies may themselves not be independent. Writing a first (or simple) syzygy

$$P_1P_3 - P_2P_4 = 0$$

as

$$S_1 = 0,$$

and other simple syzygies

$$S_2 = 0, \quad S_3 = 0, \ldots,$$

it may happen that

$$P_1' S_1 = P_2' S_2,$$

where P_1', P_2' are any products of the members of the set $P_1, P_2, \ldots P_m$.

This is termed a *second syzygy*, and necessitates a term

$$+ P_1' P_1 P_3$$

in the numerator of the generating function; if we did not get this the term $P_1' P_1 P_3$ would be omitted from the summation.

In the same manner we may have third, fourth, etc., syzygies. It follows that when the sum is presented in the form

$$\frac{1 - \{Q_1^{(1)} + Q_1^{(2)} + \ldots\} + \{Q_2^{(1)} + Q_2^{(2)} + \ldots\} - \{Q_3^{(1)} + Q_3^{(2)} + \ldots\} + \ldots}{(1 - P_1)(1 - P_2)(1 - P_3) \ldots (1 - P_m)},$$

$P_1, P_2, \ldots P_m$ denote the m ground or fundamental solutions,
$Q_1^{(1)}, Q_1^{(2)}, \ldots$ „ first syzygies,
$Q_2^{(1)}, Q_2^{(2)}, \ldots$ „ second syzygies,
$Q_3^{(1)}, Q_3^{(2)}, \ldots$ „ third syzygies,
....

and we have what is termed a complete syzygetic theory of the solution of the proposed Diophantine inequalities.

356. Some examples will now be given of simple syzygetic theories.

Consider the system of inequalities

$$\alpha_1 \geqslant \alpha_2,$$
$$\alpha_1 \geqslant \alpha_3.$$

We find

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} = \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda\mu X_1)\left(1 - \frac{1}{\lambda} X_2\right)\left(1 - \frac{1}{\mu} X_3\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1 - \mu X_1)(1 - \mu X_1 X_2)\left(1 - \frac{1}{\mu} X_3\right)}$$

$$\left(\begin{array}{c}\text{which for the inequality } \alpha_1 + \alpha_2 \geqslant \alpha_3 \text{ denotes the sum} \\ \Sigma X_1^{\alpha_1 + \alpha_2} X_2^{\alpha_2} X_3^{\alpha_3}\end{array}\right)$$

$$= \frac{1 - X_1^2 X_2 X_3}{(1 - X_1)(1 - X_1 X_2)(1 - X_1 X_3)(1 - X_1 X_2 X_3)},$$

a result which shews that there are four fundamental solutions, viz.

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 0,$$
$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 0,$$
$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 1,$$
$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1.$$

The numerator term $-X_1^2X_2X_3$ indicates the single first syzygy

$$X_1 \,.\, X_1X_2X_3 - X_1X_2 \,.\, X_1X_3 = 0.$$

In fact the solution $\alpha_1 = 2, \quad \alpha_2 = 1, \quad \alpha_3 = 1,$

can be formed *either* by adding the first and fourth *or* the second and third fundamental solutions.

The general solution may be written

$$\begin{aligned} \alpha_1 &= c_1 + c_2 + c_3 + c_4, \\ \alpha_2 &= \quad\;\; c_2 \qquad\;\, + c_4, \\ \alpha_3 &= \qquad\qquad c_3 + c_4, \end{aligned}$$

c_1, c_2, c_3, c_4 being arbitrary positive integers.

The enumerating generating function is, by putting $X_1 = X_2 = X_3 = x$,

$$\frac{1 + x^2}{(1-x)(1-x^2)(1-x^3)} \equiv \frac{\mathbf{(4)}}{\mathbf{(1)\,(2)^2\,(3)}}.$$

357. Next consider the system

$$\alpha_1 \leqslant \alpha_2 \geqslant \alpha_3 \geqslant \alpha_4.$$

Using three auxiliaries λ, μ, ν we find

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4} = \underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{1}{\lambda} X_1\right)(1 - \lambda\mu X_2)\left(1 - \frac{\nu}{\mu} X_3\right)\left(1 - \frac{1}{\nu} X_4\right)}.$$

We may eliminate λ, μ, ν in any order we please, and consideration will in most cases indicate the best order to select. In the present instance the order makes little difference. Eliminating λ we find

$$\underset{\geqslant}{\Omega} \frac{1}{(1 - \mu X_1X_2)(1 - \mu X_2)\left(1 - \frac{\nu}{\mu} X_3\right)\left(1 - \frac{1}{\nu} X_4\right)},$$

(which for the inequalities $\alpha_1 + \alpha_2 \geqslant \alpha_3$, $\alpha_3 \geqslant \alpha_4$ denotes the sum $\Sigma X_1^{\alpha_1} X_2^{\alpha_1+\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4}$).

Now, eliminating ν, we find

$$\underset{\geqslant}{\Omega} \frac{1}{(1 - \mu X_1X_2)(1 - \mu X_2)\left(1 - \frac{1}{\mu} X_3\right)\left(1 - \frac{1}{\mu} X_3X_4\right)},$$

(which for the inequality $\alpha_1 + \alpha_2 \geqslant \alpha_3 + \alpha_4$ denotes the sum $\Sigma X_1^{\alpha_1} X_2^{\alpha_1+\alpha_2} X_3^{\alpha_3+\alpha_4} X_4^{\alpha_4}$).

Finally eliminating μ by a formula given *ante* Art. 348 we find

$$\frac{1 - X_1X_2^2X_3 - X_1X_2^2X_3X_4 - X_1X_2^2X_3^2X_4 + X_1X_2^3X_3^2X_4 + X_1^2X_2^3X_3^2X_4}{(1 - X_1X_2)(1 - X_2)(1 - X_1X_2X_3)(1 - X_2X_3)(1 - X_2X_3X_4)(1 - X_1X_2X_3X_4)},$$

the real generating function of the solution of the inequalities.

We have thus six fundamental solutions, viz.

$$\begin{array}{llll}
\alpha_1 = 1, & \alpha_2 = 1, & \alpha_3 = 0, & \alpha_4 = 0, \\
\alpha_1 = 0, & \alpha_2 = 1, & \alpha_3 = 0, & \alpha_4 = 0, \\
\alpha_1 = 1, & \alpha_2 = 1, & \alpha_3 = 1, & \alpha_4 = 0, \\
\alpha_1 = 0, & \alpha_2 = 1, & \alpha_3 = 1, & \alpha_4 = 0, \\
\alpha_1 = 0, & \alpha_2 = 1, & \alpha_3 = 1, & \alpha_4 = 1, \\
\alpha_1 = 1, & \alpha_2 = 1, & \alpha_3 = 1, & \alpha_4 = 1,
\end{array}$$

shewn by the denominator factors; also three first syzygies given by

$$X_2 \,.\, X_1X_2X_3 - X_1X_2 \,.\, X_2X_3 = S_1 = 0,$$
$$X_2 \,.\, X_1X_2X_3X_4 - X_1X_2 \,.\, X_2X_3X_4 = S_2 = 0,$$
$$X_2X_3 \,.\, X_1X_2X_3X_4 - X_1X_2X_3 \,.\, X_2X_3X_4 = S_3 = 0,$$

and two second syzygies given by

$$X_2X_3X_4 \,.\, X_1X_2^2X_3 - X_2X_3 \,.\, X_1X_2^2X_3X_4 = 0,$$
$$X_1X_2X_3 \,.\, X_1X_2^2X_3X_4 - X_1X_2 \,.\, X_1X_2^2X_3^2X_4 = 0.$$

The general solution of the inequalities is

$$\begin{aligned}
\alpha_1 &= c_1 \phantom{{}+c_2} + c_3 \phantom{{}+c_4+c_5} + c_6, \\
\alpha_2 &= c_1 + c_2 + c_3 + c_4 + c_5 + c_6, \\
\alpha_3 &= \phantom{c_1+c_2+{}} c_3 + c_4 + c_5 + c_6, \\
\alpha_4 &= \phantom{c_1+c_2+c_3+c_4+{}} c_5 + c_6.
\end{aligned}$$

358. The reader who is interested in the subject of linear homogeneous Diophantine analysis should, as regards equalities, study a valuable paper by E. B. Elliott*. It has been shewn in Art. 350 that every Diophantine Inequality is a particular case of a Diophantine Equality, so that there is necessarily much that is common to the two theories. They are not however quite parallel because the Inequality, being a particular case, lends itself to particular treatment which is not available for the Equality in general. Thus although Elliott gives a discussion of the Equality

$$a\alpha = b\beta + c\gamma,$$

he does not make the solution depend upon the ordinary theory of continued fractions because this dependence only arises, for the particular case $c = 1$, when the Equality merges into the Inequality

$$a\alpha \geqslant b\beta,$$

which is fully discussed in a subsequent chapter of this work.

* "On Linear Homogeneous Diophantine Equations," *Quarterly Journal of Pure and Applied Mathematics*, No. 136, 1903.

Some examples of his method are now given in the notation employed in this book.

Taking the Equality to be

$$A_1\alpha_1 + A_2\alpha_2 + \ldots + A_i\alpha_i$$
$$= B_1\beta_1 + B_2\beta_2 + \ldots + B_j\beta_j,$$

where the numerical magnitudes A, B are integers which have no common divisor and the magnitudes α, β are integers to be determined. He shews that

$$\Sigma x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_i^{\alpha_i} x_{i+1}^{\beta_1} \ldots x_{i+j}^{\beta_j}$$

is expressible in the form

$$\underset{=}{\Omega}\frac{1}{(1-\lambda^{A_1}x_1)(1-\lambda^{A_2}x_2)\ldots(1-\lambda^{A_i}x_i)\,.\,(1-\lambda^{-B_1}x_{i+1})\ldots(1-\lambda^{-B_j}x_{i+j})},$$

with an auxiliary quantity λ.

To reduce this expression he makes use of the equation

$$\frac{1}{(1-x_s\lambda^{A_s})(1-x_t\lambda^{-B_t})} = \frac{1}{1-x_s x_t\lambda^{A_s-B_t}}\left(\frac{1}{1-x_s\lambda^{A_s}} + \frac{1}{1-x_t\lambda^{-B_t}} - 1\right),$$

choosing A_s and B_t to be the greatest of the quantities A, B respectively. The generating function is thus expressed as the sum of three fractions, each with either ± 1 for numerator. Each of the three fractions is on the whole simpler than the original. The process is continued with each fraction but it cannot be indefinitely applied. Eventually the original fraction will be replaced by a sum of fractions each with either ± 1 for numerator, in none of which is there in the denominator *both* positive and negative powers of λ. The factors of a denominator will either involve both factors without λ and with positive powers of λ *or* both factors without λ and with negative powers of λ. No single denominator will involve *both* positive and negative powers of λ. Putting all factors, which involve positive or negative powers of λ, equal to unity in these fractions gives the completion of the operation $\underset{=}{\Omega}$ and the remaining denominator factors indicate the fundamental solution of the equality.

As an example consider the Equality

$$3\alpha_1 = \beta_1 + 5\beta_2.$$

The crude generating function

$$\underset{=}{\Omega}\frac{1}{(1-\lambda^3 x_1)(1-\lambda^{-1}x_2)(1-\lambda^{-5}x_3)}$$

is reduced by the method to the expression

$$\frac{1}{(1-x_1x_2^3)(1-x_1^2x_2x_3)} + \frac{1}{(1-x_1^5x_3^3)(1-x_1^2x_2x_3)} - \frac{1}{1-x_1^2x_2x_3},$$

indicating the fundamental solutions

$$(\alpha_1, \beta_1, \beta_2) = (1, 3, 0), (5, 0, 3), (2, 1, 1).$$

The Equality considered is in fact equivalent to the Inequality

$$3\alpha_1 \geqslant 5\beta_2,$$

and is fully considered in the subsequent chapter to which reference has been made above.

359. In particular consider the Equality

$$A\alpha = B_1\beta_1 + B_2\beta_2 + \ldots + B_j\beta_j,$$

for which the crude function is

$$\underset{=}{\Omega} \frac{1}{(1-\lambda^A x_1)(1-\lambda^{-B_1}x_2)(1-\lambda^{-B_2}x_3)\ldots(1-\lambda^{-B_j}x_{j+1})}.$$

First when $A = 1$, suppose that

$$G(z) = b_0 + b_1 z + b_2 z^2 + \ldots,$$

a power series free from negative powers of z. We take $G(z)$ to be the series itself or the function of which it is the formal expansion.

Write the function under the operation $\underset{=}{\Omega}$

$$\frac{1}{1-\lambda x_1} G(\lambda^{-1}),$$

which is $$(1 + \lambda x_1 + \lambda^2 x_1^2 + \ldots)(b_0 + b_1\lambda^{-1} + b_2\lambda^{-2} + \ldots),$$

shewing that $$\underset{=}{\Omega} \frac{1}{1-\lambda x_1} G(\lambda^{-1}) = G(x_1),$$

and that the generating function is

$$\frac{1}{(1-x_1^{B_1}x_2)(1-x_1^{B_2}x_3)\ldots(1-x_1^{B_j}x_{j+1})}.$$

Secondly when $A > 1$, the terms free from λ in

$$\frac{1}{1-\lambda^A x_1} G(\lambda^{-1}),$$

are $$1 + x_1 b_\lambda + x_1^2 b_{2\lambda} + \ldots,$$

which has the expression

$$\frac{1}{A} \sum_\rho G(\rho x_1^{\frac{1}{A}}),$$

where $\rho^A = 1$ and the summation is for each root ρ of this equation.

This however is not a convenient solution. It is better to multiply numerator and denominator of $G(\lambda^{-1})$ by such factors that the denominator becomes an integral function of λ^{-A}. For we can then write λ for λ^A and the case can be dealt with in the same manner as when $A = 1$.

For example—for the Equality

$$6\alpha = 2\beta_1 + 3\beta_2 + 12\beta_3$$

we have
$$\underset{=}{\Omega} \frac{1}{(1-\lambda^6 x_1)(1-\lambda^{-2}x_2)(1-\lambda^{-3}x_3)(1-\lambda^{-12}x_4)},$$

which we may write

$$\underset{=}{\Omega} \frac{(1+\lambda^{-2}x_2+\lambda^{-4}x_2^2)(1+\lambda^{-3}x_3)}{(1-\lambda^6 x_1)(1-\lambda^{-6}x_2^3)(1-\lambda^{-6}x_3^2)(1-\lambda^{-12}x_4)},$$

wherein we may put the numerator function equal to unity because the remaining terms plainly contribute nothing to the terms of the whole expression that are free from λ. Doing this and then putting λ for λ^6 we find

$$\underset{=}{\Omega} \frac{1}{(1-\lambda x_1)(1-\lambda^{-1}x_2^3)(1-\lambda^{-1}x_3^2)(1-\lambda^{-2}x_4)}$$

$$= \frac{1}{(1-x_1x_2^3)(1-x_1x_3^2)(1-x_1^2x_4)},$$

indicating the fundamental solutions

$$(\alpha, \beta_1, \beta_2, \beta_3) = (1, 3, 0, 0), (1, 0, 2, 0), (2, 0, 0, 1).$$

In the above process after multiplication of numerator and denominator by certain factors we retain in the numerator only terms which involve integral powers of λ^{-6}.

One particular case is for the Equality

$$A\alpha_1 = mA\beta_1 + P\beta_2,$$

where P is prime to A.

Elliott gives the generating function

$$\frac{1}{(1-x_1^m x_2)(1-x_1^P x_3^A)}$$

and the reader will find that it is at once established by applying the above principle to the crude function

$$\underset{=}{\Omega} \frac{1}{(1-\lambda^A x_1)(1-\lambda^{-mA}x_2)(1-\lambda^{-P}x_3)}.$$

CHAPTER III

THE DIOPHANTINE INEQUALITY $a\alpha \geqslant b\beta$

360. The Inequality is equivalent to the Equality $a\alpha = b\beta + \gamma$.

The problem is to obtain the fundamental solutions by forming the sums $\Sigma x^{\alpha} y^{\beta}$, $\Sigma x^{\alpha} y^{\beta} z^{a\alpha - b\beta}$ for every pair of numerical magnitudes α, β which satisfy the inequality.

If
$$\overset{\geqslant}{E}$$
denotes the smallest integer which $\geqslant$ the quantity which follows it, we may carry out the summation by first summing α from $\overset{\geqslant}{E}\frac{b}{a}$ to ∞ and then summing β from 0 to ∞. The result of this procedure is

$$\frac{1 + x^{E\frac{b}{a}} y + x^{E\frac{2b}{a}} y^2 + \ldots + x^{E\frac{a-1}{a}b} y^{a-1}}{(1-x)(1-x^b y^a)},$$

which certainly proves that the fundamental solutions are *included in* the exponents of the x, y products in numerator and denominator; but it does not specify them. The true method leads to a sum of algebraic fractions and is essentially connected with the Theory of Continued Fractions.

We have
$$\Sigma x^{\alpha} y^{\beta} = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^a x)\left(1 - \frac{1}{\lambda^b} y\right)},$$

and we suppose a, b to be relatively prime positive integers.

We require two lemmas.

361. *Lemma I.* "The relation

$$a\alpha \geqslant b\beta$$

may be made to depend upon a similar relation in which a is unchanged and $a > b$."

For suppose $a \ngtr b$; if p be the greatest integer in $\frac{b}{a}$ the given relation implies the relation

$$\alpha \geqslant p\beta,$$

and introducing a second auxiliary we may write

$$\Sigma x^{\alpha} y^{\beta} = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a}\mu x)\left(1-\frac{1}{\lambda^{b}\mu^{p}}y\right)}.$$

Eliminating μ we find

$$\Sigma x^{\alpha} y^{\beta} = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a} x)\left(1-\frac{x^{p}y}{\lambda^{b-pa}}\right)},$$

which denotes the sum

$$\Sigma x^{\alpha}(x^{p}y)^{\beta} = \Sigma x^{\alpha+p\beta} y^{\beta},$$

where α, β are connected by the inequality

$$a\alpha \geqslant (b-pa)\beta.$$

We have therefore reduced the sum

$\Sigma x^{\alpha} y^{\beta}$, for the relation $a\alpha \geqslant b\beta$

to the sum $\Sigma x^{\alpha+p\beta} y^{\beta}$ „ „ $a\alpha \geqslant (b-pa)\beta$.

This proves the lemma.

362. *Lemma II.* "The relation

$$a\alpha \geqslant b\beta,$$

where $a > b$, may be made to depend upon a similar relation in which b is unchanged and $a < b$."

The relation $a\alpha \geqslant b\beta$, in which $a > b$, may be broken up into the two simultaneous sets of relations

$$\begin{array}{ll} a\alpha \geqslant b\beta, & a\alpha \geqslant b\beta, \\ \beta \geqslant \alpha, & \alpha > \beta, \end{array}$$

and the second simultaneous set

$$\begin{array}{l} a\alpha \geqslant b\beta, \\ \alpha > \beta, \end{array}$$

may be replaced by the single relation

$$\alpha > \beta$$

because, a being $> b$, the relation

$$a\alpha \geqslant b\beta$$

is implied thereby.

We therefore separate a portion of the sum corresponding to $\alpha > \beta$, viz.

$$\underset{\geqslant}{\Omega} \frac{\frac{1}{\lambda}}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} \quad \text{or} \quad \frac{x}{(1-x)(1-xy)},$$

and consider the remaining portion due to the first set

$$a\alpha \geqslant b\beta,$$
$$\beta \geqslant \alpha.$$

This portion is $$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda^a}{\mu}x\right)\left(1-\frac{\mu}{\lambda^b}y\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-b}xy)\left(1-\frac{1}{\lambda^b}y\right)},$$

which gives the sum $\Sigma (xy)^\alpha y^\beta$ or $\Sigma x^\alpha y^{\alpha+\beta}$

for the relation $(a-b)\,\alpha \geqslant b\beta.$

If $a-b<b$ we have done what was required, but if $a-b>b$ we may repeat the process; and just as we have found

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^a x)\left(1-\frac{1}{\lambda^b}y\right)} = \frac{x}{(1-x)(1-xy)} + \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-b}xy)\left(1-\frac{1}{\lambda^b}y\right)},$$

we shall find that

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-b}xy)\left(1-\frac{1}{\lambda^b}y\right)} = \frac{xy}{(1-xy)(1-xy^2)} + \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-2b}xy^2)\left(1-\frac{1}{\lambda^b}y\right)}.$$

Moreover if q be the greatest integer in $\frac{a}{b}$ we obtain finally

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^a x)\left(1-\frac{1}{\lambda^b}y\right)}$$

$$= \frac{x}{(1-x)(1-xy)} + \frac{xy}{(1-xy)(1-xy^2)} + \dots + \frac{xy^{q-1}}{(1-xy^{q-1})(1-xy^q)}$$

$$+ \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-qb}xy^q)\left(1-\frac{1}{\lambda^b}y\right)},$$

where the last written portion of the sum represents the sum

$$\Sigma (xy^q)^\alpha y^\beta \quad \text{or} \quad \Sigma x^\alpha y^{q\alpha+\beta}$$

for the relation $$(a - qb)\,\alpha \geqslant b\beta.$$

This proves the second lemma.

363. The two lemmas evidently yield a process of reduction which can be pushed to the last extent.

In what follows a will be considered $> b$. If $b > a$ it will merely be necessary to write $b - pa$ for b and $x^p y$ for y where p is the greatest integer in $\frac{b}{a}$.

The reduction depends upon the convergents of the continued fraction to $\frac{a}{b}$.

Let $$\frac{a}{b} = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\cdots\frac{1}{+ a_n},$$

where n is uneven; this is always possible because a_n may be written

$$(a_n - 1) + \frac{1}{1}.$$

Take as the first two convergents

$$\frac{0}{1} = \frac{p_{-1}}{q_{-1}}, \quad \frac{1}{0} = \frac{p_0}{q_0},$$

and write down the convergents of uneven order

$$\frac{p_{-1}}{q_{-1}}, \quad \frac{p_1}{q_1}, \quad \frac{p_3}{q_3}, \ldots \frac{p_n}{q_n}.$$

Now form the ascending series of intermediate convergents, viz.

$$\frac{p_{-1}}{q_{-1}}, \quad \frac{p_{-1}+p_0}{q_{-1}+q_0}, \quad \frac{p_{-1}+2p_0}{q_{-1}+2q_0}, \ldots \frac{p_{-1}+(a_1-1)\,p_0}{q_{-1}+(a_1-1)\,q_0},$$

$$\frac{p_1}{q_1}, \quad \frac{p_1+p_2}{q_1+q_2}, \quad \frac{p_1+2p_2}{q_1+2q_2}, \ldots \frac{p_1+(a_3-1)\,p_2}{q_1+(a_3-1)\,q_2},$$

....

$$\frac{p_{n-2}}{q_{n-2}}, \quad \frac{p_{n-2}+p_{n-1}}{q_{n-2}+q_{n-1}}, \quad \frac{p_{n-2}+2p_{n-1}}{q_{n-2}+2q_{n-1}}, \ldots \frac{p_{n-2}+(a_n-1)\,p_{n-1}}{q_{n-2}+(a_n-1)\,q_{n-1}},$$

where of course $\frac{p_n}{q_n} = \frac{a}{b}$.

Applying now the second lemma to the crude generator

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^a x)\left(1-\frac{1}{\lambda^b} y\right)},$$

we obtain

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-a_1 b} x y^{a_1})\left(1-\frac{1}{\lambda^b} y\right)} + \sum_1^{a_1} \frac{x y^{a_1-1}}{(1-xy^{a_1-1})(1-xy^{a_1})},$$

where in the expression last written the summation is in regard to a_1. Now

$$\Sigma \frac{x y^{a_1-1}}{(1-xy^{a_1-1})(1-xy^{a_1})}$$

$$= \frac{x^1 y^0}{(1-x^1 y^0)(1-x^{1+0} y^{0+1})} + \frac{x^{1+0} y^{0+1}}{(1-x^{1+0} y^{0+1})(1-x^{1+2.0} y^{0+2.1})}$$

$$+ \frac{x^{1+2.0} y^{0+2.1}}{(1-x^{1+2.0} y^{0+2.1})(1-x^{1+3.0} y^{0+3.1})} + \ldots$$

$$+ \frac{x^{1+(a_1-1).0} y^{0+(a_1-1).1}}{\{1-x^{1+(a_1-1)0} y^{0+(a_1-1)1}\}\{1-x^{1+a_1.0} y^{0+a_1.1}\}}$$

$$= \frac{x^{q_{-1}} y^{p_{-1}}}{(1-x^{q_{-1}} y^{p_{-1}})(1-x^{q_{-1}+q_0} y^{p_{-1}+p_0})}$$

$$+ \frac{x^{q_{-1}+q_0} y^{p_{-1}+p_0}}{(1-x^{q_{-1}+q_0} y^{p_{-1}+p_0})(1-x^{q_{-1}+2q_0} y^{p_{-1}+2p_0})}$$

$$+ \frac{x^{q_{-1}+2q_0} y^{p_{-1}+2p_0}}{(1-x^{q_{-1}+2q_0} y^{p_{-1}+2p_0})(1-x^{q_{-1}+3q_0} y^{p_{-1}+3p_0})} + \ldots$$

$$+ \frac{x^{q_{-1}+(a_1-1)q_0} y^{p_{-1}+(a_1-1)p_0}}{\{1-x^{q_{-1}+(a_1-1)q_0} y^{p_{-1}+(a_1-1)p_0}\}\{1-x^{q_1} y^{p_1}\}},$$

wherein the exponents are derived from the intermediate convergents to $\frac{p_{-1}}{q_{-1}}, \frac{p_1}{q_1}$ inclusive of these principal convergents.

364. The remaining part must now be subjected to the process of the first lemma and we find that

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-a_1 b} \mu x y^{a_1})\left(1-\frac{y}{\lambda^b \mu^{a_2}}\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda^{a-a_1 b} x^{q_1} y^{p_1})\left(1-\frac{x^{q_2} y^{p_2}}{\lambda^{p_2 b - q_2 a}}\right)}.$$

Now the *second lemma* yields

$$\underset{\geqslant}{\Omega} \frac{1}{\{1-\lambda^{a-a_1 b-a_3(p_2 b-q_2 a)}x^{q_1+a_3 q_2}y^{p_1+a_3 p_2}\}\left(1-\dfrac{x^{q_2}y^{p_2}}{\lambda^{p_2 b-q_2 a}}\right)}$$

$$+\sum_1^{a_3}\frac{x^{q_1}y^{p_1}(x^{q_2}y^{p_2})^{a_3-1}}{\{1-x^{q_1}y^{p_1}(x^{q_2}y^{p_2})^{a_3-1}\}\{1-x^{q_1}y^{p_1}(x^{q_2}y^{p_2})^{a_3}\}}$$

$$=\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda^{q_3 a-p_3 b}x^{q_3}y^{p_3})\left(1-\dfrac{x^{q_2}y^{p_2}}{\lambda^{p_2 b-q_2 a}}\right)}$$

$$+\frac{x^{q_1}y^{p_1}}{(1-x^{q_1}y^{p_1})(1-x^{q_1+q_2}y^{p_1+p_2})}+\frac{x^{q_1+q_2}y^{p_1+p_2}}{(1-x^{q_1+q_2}y^{p_1+p_2})(1-x^{q_1+2q_2}y^{p_1+2p_2})}+\dots$$

$$+\frac{x^{q_1+(a_3-1)q_2}y^{p_1+(a_3-1)p_2}}{\{1-x^{q_1+(a_3-1)q_2}y^{p_1+(a_3-1)p_2}\}(1-x^{q_3}y^{p_3})},$$

and we have before us a new portion of the generating function corresponding to the intermediate convergents to the principal convergents $\frac{p_1}{q_1}$, $\frac{p_3}{q_3}$, the latter both included.

365. The portion remaining,

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda^{q_3 a-p_3 b}x^{q_3}y^{p_3})\left(1-\dfrac{x^{q_2}y^{p_2}}{\lambda^{p_2 b-q_2 a}}\right)},$$

is now subjected to *Lemma I*, and the lemmas operate alternately until the final result is reached.

The last remaining portion must be

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda^{q_n a-p_n b}x^{q_n}y^{p_n})\left(1-\dfrac{x^{q_{n-1}}y^{p_{n-1}}}{\lambda^{p_{n-1}b-q_{n-1}a}}\right)},$$

which since $q_n a - p_n b = 0$ is

$$\frac{1}{1-x^{q_n}y^{p_n}},$$

or

$$1+\frac{x^{q_n}y^{p_n}}{1-x^{q_n}y^{p_n}}.$$

Hence the complete generating function is

$$1+\frac{x}{(1-x)(1-xy)}+\dots+\frac{x^{r_1}y^{s_1}}{(1-x^{r_1}y^{s_1})(1-x^{r_2}y^{s_2})}+\dots+\frac{x^{q_n}y^{p_n}}{1-x^{q_n}y^{p_n}},$$

wherein there is a term corresponding to every pair of consecutive members of the ascending intermediate series of convergents to $\frac{a}{b}$, viz.

$$\frac{s_1}{r_1},\ \frac{s_2}{r_2}.$$

From well-known properties of the convergents every xy product that appears in the function yields a ground or fundamental solution of the relation

$$a\alpha \geqslant b\beta.$$

Hence the ground solutions are

$$\alpha = r, \quad \beta = s,$$

where $\frac{s}{r}$ is in its lowest terms and is, at pleasure, any member of the ascending intermediate series of convergents to $\frac{a}{b}$.

366. Moreover every solution of the inequality is expressible in terms of a consecutive pair of intermediate convergent numbers.

Thus the general solution may be written

$$\alpha = c_1 r_1 + c_2 r_2, \quad \beta = c_1 s_1 + c_2 s_2,$$

where c_1, c_2 are arbitrary positive integers.

367. If we have before us the Equality

$$a\alpha = b\beta + \gamma,$$

the sum $\Sigma x^\alpha y^\beta z^\gamma$ for the whole of the solutions is formed by multiplying each xy product

$$x^r y^s$$

by

$$z^{ar-bs},$$

and the general solution is

$$\alpha = c_1 r_1 + c_2 r_2, \quad \beta = c_1 s_1 + c_2 s_2, \quad \gamma = a(c_1 r_1 + c_2 r_2) - b(c_1 s_1 + c_2 s_2).$$

Observe that we may also write

$$\gamma = c_1 (ar_1 - bs_1) + c_2 (ar_2 - bs_2).$$

368. As an example take the inequality

$$779\alpha \geqslant 207\beta,$$

where

$$\frac{779}{207} = 3 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{1}.$$

The principal convergents are

$$\frac{0}{1},\ \frac{1}{0},\ \frac{3}{1},\ \frac{4}{1},\ \frac{15}{4},\ \frac{64}{17},\ \frac{143}{38},\ \frac{636}{169},\ \frac{779}{207},$$

and the ascending intermediate series

$$\frac{0}{1},\ \frac{1}{1},\ \frac{2}{1},\ \frac{3}{1},\ \frac{7}{2},\ \frac{11}{3},\ \frac{15}{4},\ \frac{79}{21},\ \frac{143}{38},\ \frac{779}{207}.$$

If the principal convergents are $2m+3$ in number, the number in the intermediate series is

$$1 + a_1 + a_3 + a_5 + \ldots + a_{2m+1}.$$

Here $\Sigma x^\alpha y^\beta$

$$= 1 + \frac{x}{(1-x)(1-xy)} + \frac{xy}{(1-xy)(1-xy^2)} + \frac{xy^2}{(1-xy^2)(1-xy^3)}$$
$$+ \frac{xy^3}{(1-xy^3)(1-x^2y^7)} + \frac{x^2y^7}{(1-x^2y^7)(1-x^3y^{11})} + \frac{x^3y^{11}}{(1-x^3y^{11})(1-x^4y^{15})}$$
$$+ \frac{x^4y^{15}}{(1-x^4y^{15})(1-x^{21}y^{79})} + \frac{x^{21}y^{79}}{(1-x^{21}y^{79})(1-x^{38}y^{143})} + \frac{x^{38}y^{143}}{(1-x^{38}y^{143})(1-x^{207}y^{779})}$$
$$+ \frac{x^{207}y^{779}}{(1-x^{207}y^{779})},$$

and the fundamental solutions of the relation are

α	β	γ
1	0	779
1	1	572
1	2	365
1	3	158
2	7	109
3	11	60
4	15	11
21	79	6
38	143	1
207	779	0

γ having reference to the equality $779\alpha = 207\beta + \gamma$.

The successive pairs of values of α, β are gradually increasing approximations to the simplest solution of the equality

$$779\alpha = 207\beta.$$

369. It appears that the ordinary theory of numerical continued fractions is coincident with the simplest case of Partition Analysis.

It is a theory concerned with, in the first instance, two relative prime integers. It is clear that the fundamental solutions of simultaneous Diophantine Inequalities involving many unknowns will give properties of a set of numbers and constitute a generalization of the Theory of Continued Fractions.

370. As another example, take

$$77\alpha \geqslant 104\beta.$$

The operation of *Lemma I* shews that we have to sum

$$\Sigma x^\alpha (xy)^\beta$$

for the relation $$77\alpha \geqslant 27\beta.$$

Since $$\frac{77}{27} = 2 + \frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{3},$$

the principal convergents are

$$\frac{0}{1}, \frac{1}{0}, \frac{2}{1}, \frac{3}{1}, \frac{17}{6}, \frac{20}{7}, \frac{77}{27},$$

and the ascending intermediate series

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \frac{17}{6}, \frac{37}{13}, \frac{57}{20}, \frac{77}{27},$$

leading to the result $\Sigma x^{\alpha} (xy)^{\beta}$

$$= 1 + \frac{x}{(1-x)(1-x.xy)} + \frac{x.xy}{(1-x.xy)(1-x.x^2y^2)} + \dots + \frac{x^{27}(xy)^{77}}{1-x^{27}(xy)^{77}}.$$

The operation of Lemma I is however not necessary, for we have merely to form the ascending intermediate series of convergents to 77/104 in connexion with the sum

$$\Sigma x^{\alpha} y^{\beta}.$$

Thus $$\frac{77}{104} = 0 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{3}.$$

The principal convergents are

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{17}{23}, \frac{20}{27}, \frac{77}{104},$$

and the intermediate series

$$\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{5}{7}, \frac{8}{11}, \frac{11}{15}, \frac{14}{19}, \frac{17}{23}, \frac{37}{50}, \frac{57}{77}, \frac{77}{104}.$$

The generating function is now at once written down.

371. For the Equality $77\alpha = 104\beta + \gamma$

we have merely to multiply each product $x^{\alpha} y^{\beta}$ by $z^{77\alpha - 104\beta}$ and then

$$\Sigma x^{\alpha} y^{\beta} z^{\gamma}$$

$$\begin{aligned} = 1 &+ \frac{xz^{77}}{(1-xz^{77})(1-x^2yz^{50})} + \frac{x^2yz^{50}}{(1-x^2yz^{50})(1-x^3y^2z^{23})} \\ &+ \frac{x^3y^2z^{23}}{(1-x^3y^2z^{23})(1-x^7y^5z^{19})} + \frac{x^7y^5z^{19}}{(1-x^7y^5z^{19})(1-x^{11}y^8z^{15})} \\ &+ \frac{x^{11}y^8z^{15}}{(1-x^{11}y^8z^{15})(1-x^{15}y^{11}z^{11})} + \frac{x^{15}y^{11}z^{11}}{(1-x^{15}y^{11}z^{11})(1-x^{19}y^{14}z^7)} \\ &+ \frac{x^{19}y^{14}z^7}{(1-x^{19}y^{14}z^7)(1-x^{23}y^{17}z^3)} + \frac{x^{23}y^{17}z^3}{(1-x^{23}y^{17}z^3)(1-x^{50}y^{37}z^2)} \\ &+ \frac{x^{50}y^{37}z^2}{(1-x^{50}y^{37}z^2)(1-x^{77}y^{57}z)} + \frac{x^{77}y^{57}z}{(1-x^{77}y^{57}z)(1-x^{104}y^{77})} + \frac{x^{104}y^{77}}{1-x^{104}y^{77}}, \end{aligned}$$

establishing the fundamental solutions

α	β	γ
1	0	77
2	1	50
3	2	23
7	5	19
11	8	15
15	11	11
19	14	7
23	17	3
50	37	2
77	57	1
104	77	0

372. The reader will notice a peculiarity in this example because if we omit the highest value of α and the lowest value of γ, the α numbers read downwards are the same as the γ numbers read upwards. He will find it interesting to investigate this coincidence.

373. It appears from the form of the generating function that *every* solution (α, β, γ) of the Equality must be derivable from at most two of the fundamental solutions, say

$$(\alpha^{(r)}, \beta^{(r)}, \gamma^{(r)}), \quad (\alpha^{(r+1)}, \beta^{(r+1)}, \gamma^{(r+1)}),$$

and these must be *consecutive* solutions, viz. we must have

$$\alpha = c_r \, \alpha^{(r)} + c_{r+1} \, \alpha^{(r+1)},$$
$$\beta = c_r \beta^{(r)} + c_{r+1} \beta^{(r+1)},$$
$$\gamma = c_r \, \gamma^{(r)} + c_{r+1} \, \gamma^{(r+1)},$$

where c_r and c_{r+1} are positive integers.

Eliminating these magnitudes we find

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(r)} & \beta^{(r)} & \gamma^{(r)} \\ \alpha^{(r+1)} & \beta^{(r+1)} & \gamma^{(r+1)} \end{vmatrix} = 0,$$

a linear relation between α, β, γ which must be identical with

$$a\alpha - b\beta - \gamma = 0;$$

hence $$\begin{vmatrix} \gamma^{(r)} & \gamma^{(r+1)} \\ \beta^{(r)} & \beta^{(r+1)} \end{vmatrix} = a, \quad \begin{vmatrix} \gamma^{(r)} & \gamma^{(r+1)} \\ \alpha^{(r)} & \alpha^{(r+1)} \end{vmatrix} = b,$$

$$\begin{vmatrix} \alpha^{(r)} & \alpha^{(r+1)} \\ \beta^{(r)} & \beta^{(r+1)} \end{vmatrix} = 1,$$

the latter verifying an elementary property of intermediate convergents. The two former may be verified from the examples given.

Thus in connexion with the Equality $77\alpha = 104\beta + \gamma$ we find

$$\begin{vmatrix} 50 & 23 \\ 1 & 2 \end{vmatrix} = 77, \qquad \begin{vmatrix} 23 & 19 \\ 3 & 7 \end{vmatrix} = 104.$$

374. To arrive at the syzygetic theory it is necessary to express the generating function as a single algebraic fraction.

Thus in the case of the Equality

$$7\alpha = 3\beta + \gamma,$$

we find $$\Sigma x^\alpha y^\beta z^\gamma = \frac{1 - x^2y^2z^8 - x^4y^7z^7 - x^4y^8z^4 + x^5y^8z^{11} + x^5y^9z^8}{(1-xz^7)(1-xyz^4)(1-xy^2z)(1-x^3y^7)},$$

where, the fundamental solutions being indicated by the denominator factors, the first syzygies are

$$S_1 = (xyz^4)^2 - (xz^7)(xy^2z) = 0,$$
$$S_2 = (xz^7)(x^3y^7) - (xyz^4)(xy^2z)^3 = 0,$$
$$S_3 = (xyz^4)(x^3y^7) - (xy^2z)^4 = 0,$$

and the second syzygies

$$(xy^2z)^3 S_1 + (xyz^4) S_2 - (xz^7) S_3 = 0,$$
$$(x^3y^7) S_1 + (xy^2z) S_2 - (xyz^4) S_3 = 0.$$

Denoting a solution by $(\alpha;\ \beta, \gamma)$ the first syzygies shew the relations

$$(1;\ 0, 7) - 2(1;\ 1, 4) + (1;\ 2, 1) = 0,$$
$$(1;\ 0, 7) - (1;\ 1, 4) - 3(1;\ 2, 1) + (3;\ 7, 0) = 0,$$
$$(1;\ 1, 4) - 4(1;\ 2, 1) + (3;\ 7, 0) = 0,$$

from which are deduced the relations

$$(1;\ 2, 1) = -(1;\ 0, 7) + 2(1;\ 1, 4),$$
$$(3;\ 7, 0) = -4(1;\ 0, 7) + 7(1;\ 1, 4).$$

It will be observed that fundamental solutions are such that every solution is expressible in terms of them by means of a linear function in which the coefficients are positive integers. If negative integers be admissible a certain number of the fundamental solutions cease to be fundamental. In the above example the fundamental solutions $(1;\ 2, 1)$ and $(3;\ 7, 0)$ are linear functions (negative coefficients being admissible) of the solutions $(1;\ 0, 7)$, $(1;\ 1, 4)$.

CHAPTER IV

THE SIMULTANEOUS DIOPHANTINE INEQUALITIES

$$a_1 \alpha \geqslant b_1 \beta$$
$$b_2 \beta \geqslant a_2 \alpha$$

375. We assume a_1, b_1 to be relatively prime integers and also a_2, b_2.

If the given relations can be satisfied by other than zero values of the magnitudes α, β we must have

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \geqslant 0.$$

Assume that $b_1 > a_1$.

The first step is to prove that the relations may be reduced to other relations in which simultaneously $a_1 > b_1$, $b_2 > a_2$.

The crude expression for $\qquad \Sigma x^\alpha y^\beta$,

the sum being for every solution of the simultaneous inequalities, is

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda^{a_1}}{\mu^{a_2}} x\right)\left(1 - \dfrac{\mu^{b_2}}{\lambda^{b_1}} y\right)}.$$

If the fraction $\dfrac{b_1}{a_1}$ be developed in the form of a continued fraction, the first step is to write

$$\frac{b_1}{a_1} = m_1 + \frac{c_2}{c_1},$$

where $c_1 = a_1$ and $c_2 < c_1$ and then $\alpha \geqslant m_1 \beta$.

The crude expression may then be written

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda^{a_1}\nu}{\mu^{a_2}} x\right)\left(1 - \dfrac{\mu^{b_2}}{\lambda^{b_1}\nu^{m_1}} y\right)},$$

which by elimination of ν is

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda^{a_1}}{\mu^{a_2}} x\right)\left(1 - \dfrac{\mu^{b_2 - m_1 a_2}}{\lambda^{b_1 - m_1 a_1}} x^{m_1} y\right)}.$$

Herein if $b_2 - m_1 a_2 < 0$, the whole expression is equal to unity, indicating that the only solution is the singular one $\alpha = \beta = 0$.

If $b_2 - m_1 a_2 = 0$, we must have $a_2 = 1$, $b_2 = m_1$, and the whole expression is

$$\underset{\geqslant}{\Omega} \frac{1}{1 - \dfrac{x^{m_1} y}{\lambda^{b_1 - m_1 a_1}}},$$

which is unity, except when $b_1 - m_1 a_1 = 0$; in this case $a_1 = 1$ the Inequalities become

$$\alpha \geqslant b_1 \beta, \quad b_1 \beta \geqslant \alpha,$$

equivalent to $$\alpha = b_1 \beta,$$

and the generating function is

$$\frac{1}{1 - x^{b_1} y}.$$

Assume therefore $b_2 - m_1 a_2 > 0$ and put $a_2 = c_1'$, $b_2 - m_1 a_2 = c_2'$ so that the expression is

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda^{c_1}}{\mu^{c_1'}} x\right)\left(1 - \dfrac{\mu^{c_2'}}{\lambda^{c_2}} x^{m_1} y\right)}.$$

If now $c_2' > c_1'$ we have effected the reduction because $c_1 > c_2$ and we have the two relations

$$c_1 \alpha \geqslant c_2 \beta, \quad c_2' \beta \geqslant c_1' \alpha$$

connected with the sum $$\Sigma x^{\alpha + m_1 \beta} y^{\beta}.$$

If $c_2' = c_1'$ or $b_2 = (m_1 + 1) a_2$, we must take $a_2 = 1$ because a_2, b_2 are relatively prime and $b_2 > a_2$.

The expression then becomes, by elimination of c_1', connected with a single inequality. Putting this aside as a case already dealt with when the single inequality was under discussion, we are left with

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda^{c_1}}{\mu^{c_1'}} x\right)\left(1 - \dfrac{\mu^{c_2'}}{\lambda^{c_2}} x^{m_1} y\right)},$$

in which $c_1 > c_2$, $c_2' < c_1'$ and the expression must be further reduced.

Since $\quad b_2 - m_1 a_2 > 0$ and $c_2' < c_1'$ or $b_2 - m_1 a_2 < a_2$,

it is clear that m_1 is the greatest integer in $\dfrac{b_2}{a_2}$, and

$$\frac{b_2}{a_2} = m_1 + \frac{c_2'}{c_1'}.$$

Write now $$\frac{b_2}{a_2} = m_1 + \frac{1}{m_2 +} \frac{c_3'}{c_2'},$$

where $m_1 + \dfrac{1}{m_2}$ is the second convergent to $\dfrac{b_2}{a_2}$ and of course $c_3' < c_2'$.

Since $$\frac{c_1'}{c_2'} = m_2 + \frac{c_3'}{c_2'},$$

we may take for the expression

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\lambda^{c_1}}{\mu^{c_1'}\nu^{m_2}} x\right)\left(1 - \frac{\mu^{c_2'}\nu}{\lambda^{c_2}} x^{m_1} y\right)},$$

from which eliminating ν we obtain

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\lambda^{c_1 - m_2 c_2}}{\mu^{c_1' - m_2 c_2'}} x^{m_1 m_2 + 1} y^{m_2}\right)\left(1 - \frac{\mu^{c_2'}}{\lambda^{c_2}} x^{m_1} y\right)}.$$

Since $\frac{b_1}{a_1} = m_1 + \frac{c_2}{c_1}$ where $c_2 < c_1$ we may write

$$\frac{c_1}{c_2} = m_2 + \frac{c_3}{c_2},$$

whenever $c_1 - m_2 c_2$ is positive, and then

$$\frac{b_1}{a_1} = m_1 + \frac{1}{m_2 +} \frac{c_3}{c_2},$$

where c_3 is *not* necessarily $< c_2$.

If c_3 be zero, c_3' must be zero also and the generator is simply

$$\frac{1}{1 - x^{m_1 m_2 + 1} y^{m_2}},$$

corresponding to the Equality

$$m_2 \alpha = (m_1 m_2 + 1)\beta.$$

If c_3 be not zero we must have

$$c_3 >= \text{ or } < c_2.$$

If $c_3 > c_2$ the reduction is complete because also $c_2' > c_3'$.

In this case $m_1 + \frac{1}{m_2}$ is the second convergent to $\frac{b_2}{a_2}$ but is not the second convergent to $\frac{b_1}{a_1}$.

If $c_3 = c_2$ we can at once eliminate λ and obtain an expression that has been already dealt with. In such a case it is clear also that m_2 simply becomes $m_2 + 1$.

If $c_3 < c_2$, $m_1 + \frac{1}{m_2}$ is the second convergent to both $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$ and a further process of reduction is necessary.

376. To sum up the results so far reached:

(i) If $$a_1 > b_1, \quad a_2 > b_2,$$
the inequalities are already in the desired form.

(ii) If $$b_2 - m_1 a_2 > a_2, \quad a_1 > b_1 - m_1 a_1,$$
$$c_2' > c_1', \quad c_1 > c_2,$$
the final reduced form is
$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\lambda^{c_1}}{\mu^{c_1'}} x\right)\left(1 - \frac{\mu^{c_2'}}{\lambda^{c_2}} x^{p_1} y^{q_1}\right)},$$
wherein $\frac{b_1}{a_1} = m_1 + \frac{c_2}{c_1}$, $p_1 = m_1$, $q_1 = 1$, and $m_1 = \frac{p_1}{q_1}$ is the first convergent to $\frac{b_1}{a_1}$; also $\frac{b_2}{a_2} = m_1 + \frac{c_2'}{c_1'}$ and m_1 is *not* the first convergent to $\frac{b_2}{a_2}$.

(iii) If m_1 is the first convergent to both $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$,
$$c_3 > c_2, \quad c_2' > c_3',$$
and the final form is
$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\mu^{c_2'}}{\lambda^{c_2}} x^{p_1} y^{q_1}\right)\left(1 - \frac{\lambda^{c_3}}{\mu^{c_3'}} x^{p_2} y^{q_2}\right)},$$
where
$$\frac{b_1}{a_1} = m_1 + \frac{1}{m_2 +} \frac{c_3}{c_2},$$
$$\frac{b_2}{a_2} = m_1 + \frac{1}{m_2 +} \frac{c_3'}{c_2'},$$
and $m_1 + \frac{1}{m_2}$ is the second convergent to $\frac{b_2}{a_2}$ but is *not* the second convergent to $\frac{b_1}{a_1}$.

When $m_1 + \frac{1}{m_2}$ is the second convergent to both $\frac{b_1}{a_1}$ and $\frac{b_2}{a_2}$ we write
$$\frac{b_1}{a_1} = m_1 + \frac{1}{m_2 +} \frac{1}{m_3 +} \frac{c_4}{c_3},$$
$$\frac{b_2}{a_2} = m_1 + \frac{1}{m_2 +} \frac{1}{m_3 +} \frac{c_4'}{c_3'},$$
wherein, if
$$c_3 > c_4, \quad c_4' > c_3',$$
$$m_1 + \frac{1}{m_2 +} \frac{1}{m_3} = \frac{p_3}{q_3}$$
is the third convergent to $\frac{b_1}{a_1}$ but is not the third convergent to $\frac{b_2}{a_2}$.

The final reduced form is then

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda^{c_3}}{\mu^{c_3'}} x^{p_2} y^{q_2}\right)\left(1-\frac{\mu^{c_4'}}{\lambda^{c_4}} x^{p_3} y^{q_3}\right)},$$

which denotes the sum

$$\Sigma\,(x^{p_2}y^{q_2})^{\alpha}\,(x^{p_3}y^{q_3})^{\beta}$$

for the relations $\qquad c_3\alpha \geqslant c_4\beta, \quad c_4'\beta \geqslant c_3'\alpha.$

But if $\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$ have the same third convergents, a further reduction is necessary.

377. We proceed in this manner to establish the following general results.

We develop $\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$ to the point where they fail to have the same convergents. We have then two cases.

CASE I. If

$$\frac{b_1}{a_1} = m_1 + \frac{1}{m_2+}\,\frac{1}{m_3+}\,\ldots\,\frac{1}{m_n+}\,\frac{c_{n+1}}{c_n},$$

$$\frac{b_2}{a_2} = m_1 + \frac{1}{m_2+}\,\frac{1}{m_3+}\,\ldots\,\frac{1}{m_n+}\,\frac{c'_{n+1}}{c_n'},$$

where $\qquad c_n > c_{n+1}, \quad c_n' < c'_{n+1},$

so that $\frac{p_n}{q_n}$ is the nth convergent to $\frac{b_1}{a_1}$ but not to $\frac{b_2}{a_2}$; the final reduced form is

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda^{c_n}}{\mu^{c_n'}} x^{p_{n-1}} y^{q_{n-1}}\right)\left(1-\frac{\mu^{c'_{n+1}}}{\lambda^{c_{n+1}}} x^{p_n} y^{q_n}\right)},$$

which represents the sum

$$\Sigma\,(x^{p_{n-1}}y^{q_{n-1}})^{\alpha}\,(x^{p_n}y^{q_n})^{\beta},$$

for the inequalities $\qquad c_n\alpha \geqslant c_{n+1}\beta,$

$$c'_{n+1}\beta \geqslant c_n'\alpha,$$

wherein $\qquad c_n > c_{n+1}, \quad c'_{n+1} > c_n'.$

CASE II. If $\qquad c_n < c_{n+1}, \quad c_n' > c'_{n+1},$

then $\frac{p_n}{q_n}$ is the nth convergent to $\frac{b_2}{a_2}$ but not to $\frac{b_1}{a_1}$; the final reduced form is

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda^{c_{n+1}}}{\mu^{c'_{n+1}}} x^{p_n} y^{q_n}\right)\left(1-\frac{\mu^{c_n'}}{\lambda^{c_n}} x^{p_{n-1}} y^{q_{n-1}}\right)},$$

which represents the sum

$$\Sigma (x^{p_n} y^{q_n})^\alpha (x^{p_{n-1}} y^{q_{n-1}})^\beta,$$

for the inequalities

$$c_{n+1}\alpha \geqslant c_n\beta,$$
$$c_n'\beta \geqslant c'_{n+1}\alpha,$$

wherein

$$c_{n+1} > c_n, \quad c_n' > c'_{n+1}.$$

We have thus shewn how in every case to reduce the given inequalities to a standard form

$$a_1\alpha \geqslant b_1\beta,$$
$$b_2\beta \geqslant a_2\alpha,$$

wherein

$$a_1 > b_1, \quad b_2 > a_2.$$

378. As an example take the simultaneous inequalities

$$64\alpha \geqslant 275\beta,$$
$$142\beta \geqslant 33\alpha.$$

We find that

$$\frac{275}{64} = 4 + \frac{1}{3+}\frac{1}{2+}\frac{5}{7},$$
$$\frac{142}{33} = 4 + \frac{1}{3+}\frac{1}{2+}\frac{4}{3}.$$

This falls under Case I and we observe that

$$\frac{4}{1} \text{ and } \frac{13}{3}$$

are the first and second convergents to both fractions, but that $\frac{30}{7}$ is the third convergent to $\frac{275}{64}$ but *not* to $\frac{142}{33}$.

The problem thus presents itself in the form of finding the sum

$$\Sigma (x^{13}y^3)^\alpha (x^{30}y^7)^\beta,$$

for the relations

$$7\alpha \geqslant 5\beta,$$
$$4\beta \geqslant 3\alpha.$$

379. We are now able to give a complete solution of the problem of finding the fundamental solutions of the Diophantine system

$$a_1\alpha \geqslant b_1\beta,$$
$$b_2\beta \geqslant a_2\alpha,$$

wherein

$$a_1 > b_1, \quad b_2 > a_2.$$

For we can break up the system into the three systems

I	II	III
$a_1\alpha \geqslant b_1\beta$	$a_1\alpha \geqslant b_1\beta$	$a_1\alpha \geqslant b_1\beta$
$b_2\beta \geqslant a_2\alpha$	$b_2\beta \geqslant a_2\alpha$	$b_2\beta \geqslant a_2\alpha$
$\beta \geqslant \alpha$	$\alpha \geqslant \beta$	$\alpha = \beta$

in such wise that if $F_1(x, y)$, $F_2(x, y)$, $F_3(x, y)$ be the generating functions concerned with the three systems respectively, the sum that we seek is

$$F_1(x, y) + F_2(x, y) - F_3(x, y),$$

for clearly when we add the systems I and II we count the system III twice over, and since α may be $>=$ or $<\beta$ so long as the other conditions are satisfied the sum must be as stated.

We can now reject as superfluous the second relation of I (because $b_2 > a_2$), the first relation of II (because $a_1 > b_1$) and the first and second relations of III. They thus become

I	II	III
$a_1\alpha \geqslant b_1\beta$	$b_2\beta \geqslant a_2\alpha$	$\alpha = \beta$
$\beta \geqslant \alpha$	$\alpha \geqslant \beta$	

Hence $$F_1(x, y) + F_2(x, y) - F_3(x, y)$$

$$= \underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\lambda^{a_1}}{\mu} x\right)\left(1 - \frac{\mu}{\lambda^{b_1}} y\right)} + \underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\mu}{\lambda^{a_2}} x\right)\left(1 - \frac{\lambda^{b_2}}{\mu} y\right)} - \frac{1}{1 - xy}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda^{a_1 - b_1} xy)\left(1 - \frac{y}{\lambda^{b_1}}\right)} + \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda^{b_2 - a_2} xy)\left(1 - \frac{x}{\lambda^{a_2}}\right)} - \frac{1}{1 - xy}.$$

We have now only to apply the theorem of the single inequality to the two $\underset{\geqslant}{\Omega}$ expressions by forming the ascending intermediate series of convergents to $\frac{a_1 - b_1}{b_1}$ and $\frac{b_2 - a_2}{a_2}$.

We can thus determine the complete generating function and by inspection ascertain the fundamental solutions of the inequalities.

380. As an example we will continue the case of the system

$$64\alpha \geqslant 275\beta, \quad 142\beta \geqslant 33\alpha,$$

which has been reduced to finding the sum

$$\Sigma (x^{13}y^3)^\alpha (x^{30}y^7)^\beta$$

for the system $$7\alpha \geqslant 5\beta, \quad 4\beta \geqslant 3\alpha.$$

The sum $\Sigma x^\alpha y^\beta$ for the system is

$$\underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda^2 xy)\left(1 - \frac{y}{\lambda^5}\right)} + \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda xy)\left(1 - \frac{x}{\lambda^3}\right)} - \frac{1}{1 - xy}.$$

The ascending intermediate series to $\frac{2}{5}$ is

$$\frac{0}{1},\ \frac{1}{3},\ \frac{2}{5},$$

and to $\frac{1}{3}$

$$\frac{0}{1},\ \frac{1}{3}.$$

Hence

$$\Sigma x^{\alpha} y^{\beta} = 1 + \frac{xy}{(1-xy)(1-x^3y^4)} + \frac{x^3y^4}{(1-x^3y^4)(1-x^5y^7)} + \frac{x^5y^7}{1-x^5y^7}$$

$$+1+\frac{xy}{(1-xy)(1-x^4y^3)} + \frac{x^4y^3}{1-x^4y^3} - \frac{1}{1-xy}$$

$$= 1 + \frac{xy}{(1-xy)(1-x^3y^4)} + \frac{x^4y^3}{(1-xy)(1-x^4y^3)} + \frac{x^3y^4}{(1-x^3y^4)(1-x^5y^7)} + \frac{x^5y^7}{1-x^5y^7}.$$

This shews that of the system

$$7\alpha \geqslant 5\beta, \quad 4\beta \geqslant 3\alpha,$$

the fundamental solutions are

$$(\alpha, \beta) = (1, 1), (4, 3), (3, 4), (5, 7).$$

Also writing $x^{13}y^3$ for x and $x^{30}y^7$ for y the sum $\Sigma x^{\alpha} y^{\beta}$ for the system

$$64\alpha \geqslant 275\beta, \quad 142\beta \geqslant 33\alpha$$

is

$$1 + \frac{x^{43}y^{10}}{(1-x^{43}y^{10})(1-x^{159}y^{37})} + \frac{x^{142}y^{33}}{(1-x^{43}y^{10})(1-x^{142}y^{33})}$$

$$+ \frac{x^{159}y^{37}}{(1-x^{159}y^{37})(1-x^{275}y^{64})} + \frac{x^{275}y^{64}}{1-x^{275}y^{64}},$$

establishing the fundamental solutions

α	β
43	10
159	37
142	33
275	64

381. Allied to this theory is that of the simultaneous equalities

$$a_1\alpha = b_1\beta + \gamma,$$
$$b_2\beta = a_2\alpha + \delta.$$

In the sum just obtained it is merely necessary to multiply each $x^{\alpha}y^{\beta}$ term by $z^{a_1\alpha - b_1\beta} u^{b_2\beta - a_2\alpha}$ in order to obtain the sum

$$\Sigma x^{\alpha} y^{\beta} z^{\gamma} u^{\delta}$$

for all solutions of the simultaneous equalities.

It will be observed that every solution is composed of at most two fundamental solutions. The latter present themselves in two sets which are derived from two series of ascending intermediate convergents and every solution is derived from two consecutive fundamental solutions of one or other of the sets.

Denoting consecutive solutions by

$$\alpha^{(r)},\ \beta^{(r)},\ \gamma^{(r)},\ \delta^{(r)};\ \alpha^{(r+1)},\ \beta^{(r+1)},\ \gamma^{(r+1)},\ \delta^{(r+1)},$$

and by c_r, c_{r+1} undetermined positive integers we have

$$\alpha = c_r\,\alpha^{(r)} + c_{r+1}\,\alpha^{(r+1)},$$
$$\beta = c_r\beta^{(r)} + c_{r+1}\beta^{(r+1)},$$
$$\gamma = c_r\,\gamma^{(r)} + c_{r+1}\,\gamma^{(r+1)},$$
$$\delta = c_r\,\delta^{(r)} + c_{r+1}\,\delta^{(r+1)},$$

whence

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(r)} & \beta^{(r)} & \gamma^{(r)} \\ \alpha^{(r+1)} & \beta^{(r+1)} & \gamma^{(r+1)} \end{vmatrix} = 0, \quad \begin{vmatrix} \beta & \gamma & \delta \\ \beta^{(r)} & \gamma^{(r)} & \delta^{(r)} \\ \beta^{(r+1)} & \gamma^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = 0,$$

and also two other relations, which are not linearly independent of these,

$$\begin{vmatrix} \alpha & \beta & \delta \\ \alpha^{(r)} & \beta^{(r)} & \delta^{(r)} \\ \alpha^{(r+1)} & \beta^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = 0, \quad \begin{vmatrix} \alpha & \gamma & \delta \\ \alpha^{(r)} & \gamma^{(r)} & \delta^{(r)} \\ \alpha^{(r+1)} & \gamma^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = 0.$$

These must be practically identical with the four relations

$$a_1\alpha = b_1\beta + \gamma, \quad (a_1b_2 - a_2b_1)\,\beta = a_2\gamma + a_1\delta,$$
$$b_2\beta = a_2\alpha + \delta, \quad (a_1b_2 - a_2b_1)\,\alpha = b_2\gamma + b_1\delta,$$

so that on comparison we find

$$\begin{vmatrix} \alpha^{(r)} & \beta^{(r)} \\ \alpha^{(r+1)} & \beta^{(r+1)} \end{vmatrix} = \pm 1, \quad \begin{vmatrix} \alpha^{(r)} & \gamma^{(r)} \\ \alpha^{(r+1)} & \gamma^{(r+1)} \end{vmatrix} = \mp b_1, \quad \begin{vmatrix} \alpha^{(r)} & \delta^{(r)} \\ \alpha^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = \pm b_2,$$

$$\begin{vmatrix} \beta^{(r)} & \gamma^{(r)} \\ \beta^{(r+1)} & \gamma^{(r+1)} \end{vmatrix} = \mp a_1, \quad \begin{vmatrix} \beta^{(r)} & \delta^{(r)} \\ \beta^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = \pm a_2, \quad \begin{vmatrix} \gamma^{(r)} & \delta^{(r)} \\ \gamma^{(r+1)} & \delta^{(r+1)} \end{vmatrix} = \pm (a_1b_2 - a_2b_1),$$

the upper or lower sign being taken according as the consecutive solutions arise from the first or the second of the given equalities.

For the equalities $64\alpha = 275\beta + \gamma$; $142\beta = 33\alpha + \delta$ the solutions are

α	β	γ	δ
43	10	2	1
159	37	1	7
142	33	13	0
275	64	0	13

of which the second and fourth arise from the first equality and the first and third from the second. Hence the results shew that

$$\begin{vmatrix} 43 & 10 \\ 159 & 37 \end{vmatrix} = \begin{vmatrix} 159 & 37 \\ 275 & 64 \end{vmatrix} = 1; \qquad \begin{vmatrix} 43 & 10 \\ 142 & 33 \end{vmatrix} = -1;$$

$$\begin{vmatrix} 43 & 2 \\ 159 & 1 \end{vmatrix} = \begin{vmatrix} 159 & 1 \\ 275 & 0 \end{vmatrix} = -275; \qquad \begin{vmatrix} 43 & 2 \\ 142 & 13 \end{vmatrix} = 275;$$

$$\begin{vmatrix} 43 & 1 \\ 159 & 7 \end{vmatrix} = \begin{vmatrix} 159 & 7 \\ 275 & 13 \end{vmatrix} = 142; \qquad \begin{vmatrix} 43 & 1 \\ 142 & 0 \end{vmatrix} = -142;$$

$$\begin{vmatrix} 10 & 2 \\ 37 & 1 \end{vmatrix} = \begin{vmatrix} 37 & 1 \\ 64 & 0 \end{vmatrix} = -64; \qquad \begin{vmatrix} 10 & 2 \\ 33 & 13 \end{vmatrix} = 64;$$

$$\begin{vmatrix} 10 & 1 \\ 37 & 7 \end{vmatrix} = \begin{vmatrix} 37 & 7 \\ 64 & 13 \end{vmatrix} = 33; \qquad \begin{vmatrix} 10 & 1 \\ 33 & 0 \end{vmatrix} = -33;$$

$$\begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 0 & 13 \end{vmatrix} = 13; \qquad \begin{vmatrix} 2 & 1 \\ 13 & 0 \end{vmatrix} = -13;$$

the value of $a_1b_2 - a_2b_1$ here being $64 \,.\, 142 - 33 \,.\, 275 = 13$. These are properties of consecutive solutions analogous to the property of successive convergents in the theory of simple continued fractions.

The next step would be apparently to work out the theory of the inequality

$$a\alpha \geqslant b\beta + c\gamma$$

This has not yet been accomplished.

CHAPTER V

ON THE FORM OF ENUMERATING FUNCTIONS

382. The problem we take up in this chapter is one of considerable interest. The enumerating generating functions that are met with at the outset in the theory of the Partition of Numbers are such as are formed by factors of the form

$$\frac{1-x^{n+s}}{1-x^s} \text{ written for brevity } \frac{(\mathrm{n}+\mathrm{s})}{(\mathrm{s})}.$$

If such a function, involving a finite number of factors, appears as an enumerating generating function it necessarily follows that it is expressible in a finite integral form. The reasoning is precisely the same as that which may be employed to shew that

$$\frac{n\,!}{s\,!\,(n-s)\,!}$$

is an integer. This follows as a matter of course directly it is shewn that the numerical magnitude in question enumerates certain combinations. In many difficult cases of enumeration of partitions it is natural, after some calculations, to attempt to conjecture the form of the generating function. In one research the function

$$\frac{(\mathrm{n}+1)}{(1)}\left\{\frac{(\mathrm{n}+2)}{(2)}\right\}^2\frac{(\mathrm{n}+3)}{(3)}$$

appeared as the enumerating function. It is therefore necessarily expressible in a finite integral form whatever be the magnitude of the positive integer n. As the research proceeded certain calculations in a more general problem might have led the investigator to conjecture, guided by analogy, that the form he was seeking was

$$\frac{(\mathrm{n}+1)}{(1)}\left\{\frac{(\mathrm{n}+2)}{(2)}\right\}^3\left\{\frac{(\mathrm{n}+3)}{(3)}\right\}^3\frac{(\mathrm{n}+4)}{(4)},$$

but it had to be rejected forthwith because it does not happen to be expressible, for all values of n, in a finite integral form.

It is therefore natural, as well as important and interesting, to see general expression of functions, formed by factors such as those her sidered, which possess the property of expressibility in a finite integra It is moreover conceivable that the study will indicate promising pa future research.

It will be seen that there is a syzygetic theory of such expressions.

383. Consider the function

$$\left\{\frac{(n+1)}{(1)}\right\}^{\alpha_1}\left\{\frac{(n+2)}{(2)}\right\}^{\alpha_2}\left\{\frac{(n+3)}{(3)}\right\}^{\alpha_3}\cdots\left\{\frac{(n+s)}{(s)}\right\}^{\alpha_s},$$

and write it, for short,

$$X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_s^{\alpha_s}.$$

We investigate the sum

$$\Sigma X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_s^{\alpha_s}$$

for every set of numerical magnitudes

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \dots\ \alpha_s,$$

which renders the expression under the sign of summation expressible in a finite integral form *for all values of the integer n.*

Let ξ_t be that factor of $1-x^t$ which, when equated to zero, yields all the primitive roots of the equation

$$1-x^t=0.$$

Then if 1, d_1, d_2, ... t be all the divisors of t,

$$1-x^t=\xi_1\xi_{d_1}\xi_{d_2}\dots\xi_t,$$

as shewn in works on Higher Algebra.

If the product $\qquad X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\dots X_s^{\alpha_s}$

be now expressed in terms of

$$\xi_1,\ \xi_2,\ \xi_3,\ \dots\ \xi_{n+s},$$

there will be a numerator and a denominator, and if the function be integral the whole of the denominator will divide out into the numerator, leaving a finite integral algebraic expression.

We have therefore to ascertain the circumstances under which every expression ξ_t will occur at least as often in the numerator as in the denominator.

For example

$$\frac{(5)(6)^2(7)}{(1)(2)^2(3)}=\frac{\xi_1\xi_5(\xi_1\xi_2\xi_3\xi_6)^2\xi_1\xi_7}{\xi_1(\xi_1\xi_2)^2\xi_1\xi_3}$$

is in this manner seen to be finite and integral.

In general we pay no attention to ξ_1 because, at sight, it occurs with equal frequency in numerator and denominator.

With regard to ξ_2 it cancels out in the case of every factor of the form

$$\frac{(n+2m)}{(2m)}$$

if n be even, and does not occur at all in the factor

$$\frac{(n+2m+1)}{(2m+1)}$$

if n be even.

But if n be uneven, it occurs α_{2m+1} times in the numerator for every numerator factor $(n+2m+1)^{\alpha_{2m+1}}$ and α_{2m} times in the denominator for every denominator factor $(2m)^{\alpha_{2m}}$.

Hence we have the single condition expressed by the Diophantine inequality

$$\alpha_1+\alpha_3+\alpha_5+\ldots \geqslant \alpha_2+\alpha_4+\alpha_6+\ldots.$$

For ξ_3 if $m \equiv 0$ mod. 3 there is no condition;

if $m \equiv 2$ mod. 3 we must have

$$\alpha_1+\alpha_4+\alpha_7+\ldots \geqslant \alpha_3+\alpha_6+\alpha_9+\ldots,$$

and if $m \equiv 1$ mod. 3

$$\alpha_2+\alpha_5+\alpha_8+\ldots \geqslant \alpha_3+\alpha_6+\alpha_9+\ldots.$$

384. Proceeding in this manner we have the conditions:

for ξ_2 $\qquad \alpha_1+\alpha_3+\alpha_5+\ldots \geqslant \alpha_2+\alpha_4+\alpha_6+\ldots$

for ξ_3 $\qquad \begin{cases} \alpha_1+\alpha_4+\alpha_7+\ldots \geqslant \alpha_3+\alpha_6+\alpha_9+\ldots \\ \alpha_2+\alpha_5+\alpha_8+\ldots \geqslant \alpha_3+\alpha_6+\alpha_9+\ldots \end{cases}$

for ξ_4 $\qquad \begin{cases} \alpha_1+\alpha_5+\alpha_9+\ldots \geqslant \alpha_4+\alpha_8+\alpha_{12}+\ldots \\ \alpha_2+\alpha_6+\alpha_{10}+\ldots \geqslant \alpha_4+\alpha_8+\alpha_{12}+\ldots \\ \alpha_3+\alpha_7+\alpha_{11}+\ldots \geqslant \alpha_4+\alpha_8+\alpha_{12}+\ldots \end{cases}$

....

....

for ξ_{s-2} $\qquad \begin{cases} \alpha_1+\alpha_s \geqslant \alpha_{s-1} \\ \alpha_2 \geqslant \alpha_{s-1} \\ \vdots \qquad \vdots \\ \alpha_{s-2} \geqslant \alpha_{s-1} \end{cases}$

for ξ_{s-1} $\qquad \begin{cases} \alpha_1 \geqslant \alpha_s \\ \alpha_2 \geqslant \alpha_s \\ \vdots \qquad \vdots \\ \alpha_{s-1} \geqslant \alpha_s \end{cases}$

$\binom{s}{2}$ in number.

385. All the conditions are included in the inequality

$$\alpha_\sigma + \alpha_{2\sigma+\tau} + \alpha_{3\sigma+2\tau} + \ldots \geqslant \alpha_{\sigma+\tau} + \alpha_{2\sigma+2\tau} + \alpha_{3\sigma+3\tau} + \ldots,$$

σ and τ being any positive integers.

In the case of the qth inequality arising from ξ_p the values of σ, τ are q and $p-q+1$ respectively.

The next step is to construct an $\underset{\geqslant}{\Omega}$ function which shall expre〈ss the〉 conditions and lead practically to the desired summation.

386. First take $s=2$; there is but one condition

$$\alpha_1 \geqslant \alpha_2,$$

and the function is
$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 X_1)\left(1-\dfrac{1}{\lambda_1}X_2\right)}$$
$$= \frac{1}{(1-X_1)(1-X_1X_2)}.$$

We may call X_1, X_1X_2 the fundamental forms, the general form being

$$X_1^{m_1+m_2} X_2^{m_2}.$$

There are no syzygies.

387. For $s=3$ the conditions are

$$\alpha_1 + \alpha_3 \geqslant \alpha_2,$$
$$\alpha_1 \geqslant \alpha_3,$$
$$\alpha_2 \geqslant \alpha_3,$$

leading to the summation formula

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 X_1)\left(1-\dfrac{\lambda_3}{\lambda_1}X_2\right)\left(1-\dfrac{\lambda_1}{\lambda_2\lambda_3}X_3\right)},$$

λ_1, λ_2, λ_3 being the auxiliaries connected with the inequalities respectively.

Eliminating λ_3 we find

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 X_1)\left(1-\dfrac{1}{\lambda_1}X_2\right)\left(1-\dfrac{1}{\lambda_2}X_2X_3\right)},$$

which denotes the sum
$$\Sigma X_1^{a_1} X_2^{a_2} (X_2X_3)^{a_3}$$
for the inequalities
$$\alpha_1 \geqslant \alpha_2, \quad \alpha_1 \geqslant \alpha_3.$$

Writing X_2X_3 for X_3 in the sum is equivalent to writing $\alpha_2+\alpha_3$ for α_2 in the inequalities.

We now eliminate λ_2 and proceed to

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 X_1)\left(1-\frac{1}{\lambda_1}X_2\right)(1-\lambda_1 X_1 X_2 X_3)},$$

which denotes the sum

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} (X_1 X_2 X_3)^{\alpha_3},$$

for the inequality $\alpha_1+\alpha_3 \geqslant \alpha_2$, so that clearly writing X_1X_3 for X_3 in the previous sum is equivalent to writing $\alpha_1+\alpha_3$ for α_1 in the connected inequality. On the whole so far we have substituted $X_1X_2X_3$ for X_1 in the sum and have changed the original three inequalities by writing $\alpha_1+\alpha_3$ for α_1 and $\alpha_2+\alpha_3$ for α_2, thus reducing them to the single inequality $\alpha_1+\alpha_3 \geqslant \alpha_2$.

Making use now of a formula on a previous page we obtain finally

$$\frac{1-X_1^2X_2^2X_3}{(1-X_1)(1-X_1X_2)(1-X_1X_2X_3)(1-X_1X_2^2X_3)},$$

the complete solution.

The denominator factors yield the fundamental forms

$$X_1X_2X_3, \quad X_1X_2^2X_3,$$

in addition to those previously met with, and the general integral form is

$$X_1^{m_1+m_2+m_3+m_4} X_2^{m_2+m_3+2m_4} X_3^{m_3+m_4}.$$

The numerator term $-X_1^2X_2^2X_3$ indicates the simple syzygy

$$X_1 . X_1X_2^2X_3 - X_1X_2 . X_1X_2X_3 = 0.$$

The form $X_1X_2X_3$ occurs in simple partition theory, whilst

$$X_1X_2^2X_3 \equiv \frac{(1-x^{n+1})(1-x^{n+2})^2(1-x^{n+3})}{(1-x)(1-x^2)^2(1-x^3)}$$

is connected with another kind of partition which is dealt with in this volume.

388. Passing to the case $s=4$, the conditions are

$$\begin{aligned} \alpha_1+\alpha_3 &\geqslant \alpha_2+\alpha_4, \\ \alpha_1+\alpha_4 &\geqslant \alpha_3, \\ \alpha_2 &\geqslant \alpha_3, \\ \alpha_1 &\geqslant \alpha_4, \\ \alpha_2 &\geqslant \alpha_4, \\ \alpha_3 &\geqslant \alpha_4, \end{aligned}$$

of which we neglect the fifth as it is implied by the third and sixth.

These lead to

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_2\lambda_4 X_1)\left(1-\frac{\lambda_3}{\lambda_1}X_2\right)\left(1-\frac{\lambda_1\lambda_5}{\lambda_2\lambda_3}X_3\right)\left(1-\frac{\lambda_2}{\lambda_1\lambda_4\lambda_5}X_4\right)}.$$

Eliminating λ_5, λ_4, λ_3 and λ_1 in order, this becomes

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_2 X_1)(1-\lambda_2 X_1 X_2)\left(1-\frac{1}{\lambda_2}X_2 X_3\right)(1-\lambda_2 X_1 X_2 X_3 X_4)},$$

and, making use of a formula on a previous page, this is

$$\frac{1-X_1^2X_2^2X_3-X_1^2X_2^2X_3^2X_4-X_1^2X_2^3X_3^2X_4+X_1^3X_2^3X_3^2X_4^1+X_1^3X_2^4X_3^3X_4^1}{(1-X_1)(1-X_1X_2)(1-X_1X_2X_3)(1-X_1X_2^2X_3)(1-X_1X_2X_3X_4)(1-X_1X_2^2X_3^2X_4)},$$

indicating the new ground forms $X_1X_2X_3X_4$, $X_1X_2^2X_3^2X_4$, both of which are known in the theory of partitions.

We have the first syzygies

$$(A)=X_1X_2 \,.\, X_1X_2X_3-X_1 \,.\, X_1X_2^2X_3=0,$$
$$(B)=X_1X_2X_3 \,.\, X_1X_2X_3X_4-X_1 \,.\, X_1X_2^2X_3^2X_4=0,$$
$$(C)=X_1X_2^2X_3 \,.\, X_1X_2X_3X_4-X_1X_2 \,.\, X_1X_2^2X_3^2X_4=0,$$

and the second syzygies

$$X_1(C)-X_1X_2(B)=0,$$
$$X_1X_2X_3(C)-X_1X_2^2X_3(B)=0.$$

389. There is no difficulty in continuing the series. The obtaining however of the reduced forms soon becomes laborious.

Moreover there is obviously a waste of energy because in each case we not only determine the new forms but also all of those that have previously presented themselves. It will be advantageous to adopt some method that for each order only causes the forms of that order to emerge. The whole of the fundamental forms so far met with, as well as all of those that have so far presented themselves in the theory of partitions, are such that if

$$X_1^{\alpha_1}X_2^{\alpha_2}\ldots X_s^{\alpha_s}$$

be the form, then $\alpha_m=\alpha_{s+1-m}$.

We will therefore for any order impose this condition which clearly eliminates all forms of lower order. It is possible that by this method some fundamental forms will escape notice, but those that are most likely to occur in the theory cannot escape detection.

For use with the added condition we put

$$X_mX_{s+1-m}=Y_m, \text{ but } X_m=Y_m \text{ when } 2m=s+1,$$

and examine the sum $\Sigma Y_1^{\alpha_1}Y_2^{\alpha_2}Y_3^{\alpha_3}\ldots.$

390. For $s=2$ we have simply

$$\Sigma Y_1^{\alpha_1}=\frac{1}{1-Y_1}=\frac{1}{1-X_1X_2}.$$

391. For $s=3$ the conditions become

$$2\alpha_1 \geqslant \alpha_2 \geqslant \alpha_1,$$

and lead to

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda_1^2}{\lambda_2} Y_1\right)\left(1-\frac{\lambda_2}{\lambda_1} Y_2\right)},$$

which on reduction is

$$\frac{1}{(1-Y_1Y_2)(1-Y_1Y_2^2)} = \frac{1}{(1-X_1X_2X_3)(1-X_1X_2^2X_3)}.$$

392. For $s=4$ we have the same conditions as for $s=3$, viz.

$$2\alpha_1 \geqslant \alpha_2 \geqslant \alpha_1,$$

and we produce the same Y function, viz.

$$\frac{1}{(1-Y_1Y_2)(1-Y_1Y_2^2)};$$

but since now $Y_1 = X_1X_4$, $Y_2 = X_2X_3$ this is

$$\frac{1}{(1-X_1X_2X_3X_4)(1-X_1X_2^2X_3^2X_4)},$$

and we have thus found the fundamental forms that have already been found by the first method. Observe that no syzygies have so far presented themselves.

393. For $s=5$ the conditions are

$$\alpha_1+\alpha_2 \geqslant \alpha_3 \geqslant \alpha_2,$$
$$2\alpha_1 \geqslant \alpha_2 \geqslant \alpha_1,$$

leading to

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1-\frac{\lambda_1\lambda_2^2}{\lambda_4} Y_1\right)\left(1-\frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} Y_2\right)\left(1-\frac{\lambda_3}{\lambda_1} Y_3\right)},$$

wherein $\quad Y_1 = X_1X_5, \quad Y_2 = X_2X_4, \quad Y_3 = X_3.$

Eliminating λ_3, λ_4 and λ_1 in order, this becomes

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_2 Y_1Y_2Y_3)\left(1-\frac{1}{\lambda_2} Y_2Y_3\right)(1-\lambda_2 Y_1Y_2Y_3^2)},$$

or

$$\frac{1-Y_1^2Y_2^3Y_3^4}{(1-Y_1Y_2Y_3)(1-Y_1Y_2Y_3^2)(1-Y_1Y_2^2Y_3^2)(1-Y_1Y_2^2Y_3^3)}$$

$$= \frac{1-X_1^2X_2^3X_3^4X_4^3X_5^2}{(1-X_1X_2X_3X_4X_5)(1-X_1X_2X_3^2X_4X_5)(1-X_1X_2^2X_3^2X_4^2X_5)(1-X_1X_2^2X_3^3X_4^2X_5)},$$

indicating the new fundamental forms

$$X_1X_2X_3X_4X_5, \quad X_1X_2X_3^2X_4X_5, \quad X_1X_2^2X_3^2X_4^2X_5, \quad X_1X_2^2X_3^3X_4^2X_5,$$

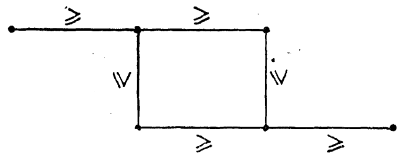
connected by the simple syzygy

$$X_1X_2X_3X_4X_5 \,.\, X_1X_2^2X_3^3X_4^2X_5 - X_1X_2X_3^2X_4X_5 \,.\, X_1X_2^2X_3^2X_4^2X_5 = 0.$$

The new forms either have been or will be met with in this work with the exception of

$$X_1X_2X_3^2X_4X_5.$$

This is associated with partitions whose parts are placed at the points of the incomplete and dislocated lattice



the magnitudes of the parts being governed by the symbols $\geqslant$ placed upon the lines between the points. The reader who has understood this chapter will have no difficulty in proving this by forming the Diophantine inequalities and examining the associated $\underset{\geqslant}{\Omega}$ sum.

394. For $s = 6$, the conditions are

$$2\alpha_2 \geqslant \alpha_1 + \alpha_3,$$
$$2\alpha_1 \geqslant \alpha_2,$$
$$\alpha_3 \geqslant \alpha_2,$$

leading to

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \frac{\lambda_2^2}{\lambda_1} Y_1\right)\left(1 - \frac{\lambda_1^2}{\lambda_2\lambda_3} Y_2\right)\left(1 - \frac{\lambda_3}{\lambda_1} Y_3\right)},$$

which quickly reduces to

$$\frac{1}{(1 - Y_1Y_2Y_3)(1 - Y_1Y_2^2Y_3^2)(1 - Y_1Y_2^2Y_3^3)},$$

yielding the fundamental forms:

$$X_1X_2X_3X_4X_5X_6,$$
$$X_1X_2^2X_3^2X_4^2X_5^2X_6,$$
$$X_1X_2^2X_3^3X_4^3X_5^2X_6,$$

unconnected by any syzygy.

The reader will notice that the results are more simple for the even orders.

395. For $s = 7$ the independent conditions are

$$2\alpha_1 + 2\alpha_3 \geqslant 2\alpha_2 + \alpha_4,$$
$$2\alpha_2 \geqslant \alpha_4 \geqslant \alpha_3 \geqslant \alpha_2 \geqslant \alpha_1,$$
$$\alpha_1 + \alpha_2 \geqslant \alpha_3,$$
$$2\alpha_1 \geqslant \alpha_2,$$

which finally lead to the function of which the numerator is

$$(1 - Y_1^2 Y_2^3 Y_3^4 Y_4^4)(1 - Y_1^2 Y_2^3 Y_3^4 Y_4^5),$$

and the denominator

$$(1 - Y_1 Y_2 Y_3 Y_4)(1 - Y_1 Y_2 Y_3 Y_4^2)(1 - Y_1 Y_2 Y_3^2 Y_4^2)$$
$$\times (1 - Y_1 Y_2^2 Y_3^2 Y_4^2)(1 - Y_1 Y_2^2 Y_3^3 Y_4^3)(1 - Y_1 Y_2^2 Y_3^3 Y_4^4),$$

establishing the fundamental forms

$$\begin{gathered} X_1 X_2 X_3 X_4 X_5 X_6 X_7, \\ X_1 X_2 X_3 X_4^2 X_5 X_6 X_7, \\ X_1 X_2 X_3^2 X_4^2 X_5^2 X_6 X_7, \\ X_1 X_2^2 X_3^2 X_4^2 X_5^2 X_6^2 X_7, \\ X_1 X_2^2 X_3^3 X_4^3 X_5^3 X_6^2 X_7, \\ X_1 X_2^2 X_3^3 X_4^4 X_5^3 X_6^2 X_7, \end{gathered}$$

connected by the simple syzygies

$$(A) = Y_1 Y_2 Y_3 Y_4 \,.\, Y_1 Y_2^2 Y_3^3 Y_4^3 - Y_1 Y_2 Y_3^2 Y_4^2 \,.\, Y_1 Y_2^2 Y_3^2 Y_4^2 = 0,$$
$$(B) = Y_1 Y_2 Y_3 Y_4 \,.\, Y_1 Y_2^2 Y_3^3 Y_4^4 - Y_1 Y_2 Y_3 Y_4^2 \,.\, Y_1 Y_2^2 Y_3^3 Y_4^3 = 0,$$

and the second or compound syzygy

$$(Y_1 Y_2 Y_3 Y_4 \,.\, Y_1 Y_2^2 Y_3^3 Y_4^4)(A) - (Y_1 Y_2 Y_3^2 Y_4^2 \,.\, Y_1 Y_2^2 Y_3^2 Y_4^2)(B) = 0.$$

396. For $s = 8$ the conditions are

$$\begin{gathered} \alpha_2 + \alpha_3 \geqslant \alpha_1 + \alpha_4, \\ \alpha_1 + \alpha_2 \geqslant \alpha_3, \\ \alpha_4 \geqslant \alpha_3 \geqslant \alpha_2, \\ 2\alpha_1 \geqslant \alpha_2, \end{gathered}$$

and the reader will have no difficulty in establishing the fundamental forms

$$\begin{gathered} X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8, \\ X_1 X_2 X_3^2 X_4^2 X_5^2 X_6^2 X_7 X_8, \\ X_1 X_2^2 X_3^2 X_4^2 X_5^2 X_6^2 X_7^2 X_8, \\ X_1 X_2^2 X_3^2 X_4^3 X_5^3 X_6^2 X_7^2 X_8, \\ X_1 X_2^2 X_3^3 X_4^3 X_5^3 X_6^3 X_7^2 X_8, \\ X_1 X_2^2 X_3^3 X_4^4 X_5^4 X_6^3 X_7^2 X_8, \end{gathered}$$

connected by three simple and two compound syzygies.

We may now conveniently take stock of the results so far obtained.

We have not so far met with any fundamental form which is not exponentially symmetrical, and the second method of investigation has effectually eliminated such from consideration.

This must not be forgotten.

All the forms that have emerged have the property that the exponent of X_1, viz. α_1, is unity. Assuming that no fundamental form exists in which $\alpha_1 = 0$ it is clear that every form, satisfying the conditions, that has $\alpha_1 = 1$ is a fundamental form.

397. A large number of the forms are seen to be such that the factors $X_1, X_2, \ldots$ can be arranged in the form of a rectangle such as

$$\begin{array}{llllll} X_1 & X_2 & X_3 & X_4 & \ldots & X_l, \\ X_2 & X_3 & X_4 & X_5 & \ldots & X_{l+1}, \\ \vdots & & & & & \\ X_m & X_{m+1} & X_{m+2} & X_{m+3} & \ldots & X_{l+m-1}, \end{array}$$

in which tableau the suffix of any X exceeds by unity the suffix of the X immediately above it or immediately to the left of it.

lm factors X are involved and the product of them all appears, from the results hitherto reached, to be a fundamental form.

The product for $m \geqslant l$ is

$$X_1 X_2^2 X_3^3 \ldots X_{l-1}^{l-1} (X_l X_{l+1} \ldots X_m)^l X_{m+1}^{l-1} X_{m+2}^{l-2} \ldots X_{m+l-2}^2 X_{m+l-1},$$

and we can establish that it is a fundamental form by simply proving that it is finite and integral for all values of the integers m, l.

This will be the next step.

Comparing the form with

$$X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_s^{\alpha_s}, \qquad s = m + l - 1,$$

it will be seen that

$$\begin{aligned} \alpha_p &= p \text{ for } l \geqslant p, \\ \alpha_p &= l \text{ for } p > l \text{ and } < m+1, \\ \alpha_{m+p} &= l - p \text{ for } l - 1 \geqslant p. \end{aligned}$$

It has been already remarked that all the Diophantine conditions may be resumed in the single formula

$$\alpha_\sigma + \alpha_{2\sigma+\tau} + \alpha_{3\sigma+2\tau} + \ldots \geqslant \alpha_{\sigma+\tau} + \alpha_{2\sigma+2\tau} + \alpha_{3\sigma+3\tau} + \ldots,$$

σ and τ being any integers.

We must first find how many terms of the series

$$\alpha_\sigma + \alpha_{2\sigma+\tau} + \alpha_{3\sigma+2\tau} + \ldots$$

appear in the first portion

$$X_1 X_2^2 X_3^3 \ldots X_{l-1}^{l-1}$$

of the form.

If the first k terms appear, then

$$k\sigma + (k-1)\tau \leqslant l - 1,$$

or

$$k \leqslant \frac{l + \tau - 1}{\sigma + \tau};$$

we may therefore take k to be the greatest integer contained in the numerical fraction

$$\frac{l+\tau-1}{\sigma+\tau},$$

and write it $I_1\dfrac{l+\tau-1}{\sigma+\tau}$ or I_1 for brevity.

Again if the first k terms appear in the first two portions

$$X_1 X_2^2 \ldots X_{l-1}^{l-1}(X_l X_{l+1}\ldots X_m)^l,$$

then $$k\sigma+(k-1)\tau \leqslant m,$$

or k is the greatest integer in

$$\frac{m+\tau}{\sigma+\tau},$$

and we write it $I_2\dfrac{m+\tau}{\sigma+\tau}$ or I_2 for brevity.

Again for the whole form

$$k\sigma+(k-1)\tau=m+l-1,$$

and k may be written $I_3\dfrac{m+l+\tau-1}{\sigma+\tau}$ or I_3 for brevity.

We can now sum the series

$$\alpha_\sigma+\alpha_{2\sigma+\tau}+\alpha_{3\sigma+2\tau}+\ldots$$

for the given product.

We first take I_1 terms, remembering that each α is here numerically equal to its suffix. The sum is

$$\tfrac{1}{2}I_1(\sigma+I_1\sigma+I_1\tau-\tau),$$

or $$\tfrac{1}{2}I_1^2(\sigma+\tau)+\tfrac{1}{2}I_1(\sigma-\tau).$$

We now take the next I_2-I_1 terms, remembering that here each α is numerically equal to l. This is clearly

$$(I_2-I_1)\,l.$$

Finally we take the next I_3-I_2 terms, remembering that each α, viz. α_p, is equal to $l+m-p$. The series to be summed is

$$\alpha_{(I_2+1)\sigma+I_2\tau}+\alpha_{(I_2+2)\sigma+(I_2+1)\tau}+\ldots+\alpha_{I_3\sigma+(I_3-1)\tau},$$

or

$$(I_3-I_2)(l+m)-\{(I_2+1)\sigma+I_2\tau+(I_2+2)\sigma+(I_2+1)\tau+\ldots+I_3\sigma+(I_3-1)\tau\}$$

or $$(I_3-I_2)(l+m)-\tfrac{1}{2}(I_3-I_2)\{(I_2+I_3+1)\sigma+(I_2+I_3-1)\tau\}.$$

Assembling the three portions of the sum and simplifying we find

$$\begin{aligned}&\alpha_\sigma+\alpha_{2\sigma+\tau}+\alpha_{3\sigma+2\tau}+\ldots\\&=\tfrac{1}{2}(I_1^2+I_2^2-I_3^2)(\sigma+\tau)\\&-\tfrac{1}{2}I_1(2l-\sigma+\tau)-\tfrac{1}{2}I_2(2m-\sigma+\tau)+\tfrac{1}{2}I_3(2l+2m-\sigma+\tau).\end{aligned}$$

Next in regard to the series

$$\alpha_{\sigma+\tau} + \alpha_{2\sigma+2\tau} + \alpha_{3\sigma+3\tau} + \dots$$

we have, analogous to I_1, I_2, I_3,

$$J_1 \frac{l-1}{\sigma+\tau} = J_1 = \text{greatest integer contained in } \frac{l-1}{\sigma+\tau},$$

$$J_2 \frac{m}{\sigma+\tau} = J_2 \quad \text{,,} \quad \text{,,} \quad \frac{m}{\sigma+\tau},$$

$$J_3 \frac{l+m-1}{\sigma+\tau} = J_3 \quad \text{,,} \quad \text{,,} \quad \frac{l+m-1}{\sigma+\tau},$$

and we find for the sum of the series

$$\alpha_{\sigma+\tau} + \alpha_{2\sigma+2\tau} + \alpha_{3\sigma+3\tau} + \dots,$$

the three portions

$$\tfrac{1}{2} J_1^2 (\sigma+\tau) + \tfrac{1}{2} J_1 (\sigma+\tau),$$

$$(J_2 - J_1)\, l,$$

$$(J_3 - J_2)(l+m) - \tfrac{1}{2}(J_3 - J_2)(J_2 + J_3 + 1)(\sigma+\tau).$$

Adding and simplifying we find

$$\begin{aligned}&\tfrac{1}{2}(J_1^2 + J_2^2 - J_3^2)(\sigma+\tau)\\ &- \tfrac{1}{2} J_1 (2l - \sigma - \tau) - \tfrac{1}{2} J_2 (2m - \sigma - \tau) + \tfrac{1}{2} J_3 (2l + 2m - \sigma - \tau).\end{aligned}$$

The general Diophantine inequality therefore assumes the form

$$\begin{aligned}&\tfrac{1}{2}(I_1^2 + I_2^2 - I_3^2)(\sigma+\tau)\\ &- \tfrac{1}{2} I_1 (2l - \sigma + \tau) - \tfrac{1}{2} I_2 (2m - \sigma + \tau) + \tfrac{1}{2} I_3 (2l + 2m - \sigma + \tau)\\ &\geqslant \tfrac{1}{2}(J_1^2 + J_2^2 - J_3^2)(\sigma+\tau)\\ &- \tfrac{1}{2} J_1 (2l - \sigma - \tau) - \tfrac{1}{2} J_2 (2m - \sigma - \tau) + \tfrac{1}{2} J_3 (2l + 2m - \sigma - \tau),\end{aligned}$$

and we must shew that it is satisfied by the values of I_1, I_2, I_3, J_1, J_2, J_3 given above in terms of l, m, σ, τ.

Since I_1 is the greatest integer in $\dfrac{l+\tau-1}{\sigma+\tau}$ and J_1 the greatest integer in $\dfrac{l-1}{\sigma+\tau}$ it is clear that I_1 is either equal to J_1 or to $J_1 + 1$.

Similarly I_2 must be either equal to J_2 or to $J_2 + 1$ and I_3 to J_3 or to $J_3 + 1$.

Also $I_1 + I_2 = I_1 \dfrac{l+\tau-1}{\sigma+\tau} + I_2 \dfrac{m+\tau}{\sigma+\tau}$ must be either equal to

$$I_3 \frac{l+m+\tau-1}{\sigma+\tau} \quad \text{or to} \quad I_3 + 1,$$

and $J_1 + J_2 = J_1 \dfrac{l-1}{\sigma+\tau} + J_2 \dfrac{m}{\sigma+\tau}$ must be either equal to

$$J_3 \frac{l+m-1}{\sigma+\tau} \quad \text{or to} \quad J_3 - 1.$$

We have therefore the relations

$$\begin{aligned} I_1 &= J_1 \text{ or } J_1+1, & I_1+I_2 &= I_3 \text{ or } I_3+1,\\ I_2 &= J_2 \text{ or } J_2+1, & J_1+J_2 &= J_3 \text{ or } J_3-1,\\ I_3 &= J_3 \text{ or } J_3+1. \end{aligned}$$

All of these alternatives are not simultaneously possible; for instance if $I_1 = J_1$, $I_2 = J_2$, $I_3 = J_3$, we cannot have $I_1 + I_2 = I_3 + 1$ because that would lead to $J_1 + J_2 = J_3 + 1$, a relation which is barred.

There are just ten possible sets of alternatives, viz.:

Case I

$$I_1 = J_1,\ I_2 = J_2,\ I_3 = J_3;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3.$$

Substituting in the inequality we find

$$I_3 \geqslant I_1 + I_2,$$

which is satisfied because $I_1 + I_2 = I_3$.

Case II

$$I_1 = J_1 + 1,\ I_2 = J_2,\ I_3 = J_3;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3 - 1$$

leads to

$$J_1 \geqslant \frac{l-\sigma-\tau}{\sigma+\tau},$$

and J_1 being the greatest integer in $\dfrac{l-1}{\sigma+\tau}$, where $\sigma+\tau$ is at least unity, the relation is satisfied.

Case III

$$I_1 = J_1 + 1,\ I_2 = J_2,\ I_3 = J_3;\quad I_1 + I_2 = I_3 + 1,\ J_1 + J_2 = J_3$$

leads to

$$J_1 \geqslant \frac{l-\sigma}{\sigma+\tau},$$

satisfied because $\sigma \geqslant 1$.

Case IV

$$I_1 = J_1,\ I_2 = J_2 + 1,\ I_3 = J_3;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3 - 1$$

leads to

$$J_2 \geqslant \frac{m}{\sigma+\tau} - 1,$$

which is true by the definition of J_2.

Case V

$$I_1 = J_1,\ I_2 = J_2 + 1,\ I_3 = J_3;\quad I_1 + I_2 = I_3 + 1,\ J_1 + J_2 = J_3$$

leads to
$$J_2 \geqslant \frac{m - \sigma}{\sigma + \tau},$$
which is obviously satisfied.

Case VI

$$I_1 = J_1 + 1,\ I_2 = J_2 + 1,\ I_3 = J_3;\quad I_1 + I_2 = I_3 + 1,\ J_1 + J_2 = J_3$$

leads to
$$J_3 \geqslant \frac{l + m - \sigma}{\sigma + \tau},$$
which is clearly satisfied.

Case VII

$$I_1 = J_1 + 1,\ I_2 = J_2,\ I_3 = J_3 + 1;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3$$

leads to
$$J_2 \geqslant \frac{m}{\sigma + \tau},$$
which is right.

Case VIII

$$I_1 = J_1,\ I_2 = J_2 + 1,\ I_3 = J_3 + 1;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3$$

leads to
$$J_1 \geqslant \frac{l}{\sigma + \tau},$$
which is satisfied.

Case IX

$$I_1 = J_1 + 1,\ I_2 = J_2 + 1,\ I_3 = J_3 + 1;\quad I_1 + I_2 = I_3,\ J_1 + J_2 = J_3 - 1$$

leads to a relation of equality between the two sides.

Case X

$$I_1 = J_1 + 1,\ I_2 = J_2 + 1,\ I_3 = J_3 + 1;\quad I_1 + I_2 = I_3 + 1,\ J_1 + J_2 = J_3$$

leads to
$$\sigma + \tau \geqslant 0,$$
which is correct.

Hence the inequality is in every case satisfied.

398. This proves that the form

$$X_1 X_2^2 X_3^3 \ldots X_{l-1}^{l-1} (X_l X_{l+1} \ldots X_m)^l X_{m+1}^{l-1} X_{m+2}^{l-2} \ldots X_{l+m-2}^2 X_{l+m-1}$$

is finite and integral and also a fundamental form.

This form plays an important rôle in a later section of this work.

It may be regarded as a First Standard Form.

The fundamental forms, that have emerged so far, not included in the tandard are

$$X_1 X_2 X_3^2 X_4 X_5,$$
$$X_1 X_2 X_3 X_4^2 X_5 X_6 X_7,$$
$$X_1 X_2 X_3^2 X_4^2 X_5^2 X_6 X_7,$$
$$X_1 X_2 X_3^2 X_4^2 X_5^2 X_6^2 X_7 X_8,$$
$$X_1 X_2^2 X_3^2 X_4^3 X_5^3 X_6^2 X_7^2 X_8,$$

and it may be desired to obtain information concerning them.

In the standard form if $\alpha_p = \alpha_{p+1}$ no exponent α_{p+q} is greater than α_p. We may therefore eliminate the standard forms from consideration by imposing other Diophantine relations such as

$$\alpha_p = \alpha_{p+1} < \alpha_{p+2}$$

upon the conditions to which the exponents are subject.

In this way there is no doubt that other Standard Forms can be obtained, but for the purpose of this work it is not necessary to carry the investigation any further.

CHAPTER VI

ON THE ALGEBRAIC FORMS OF INTEGERS

399. We may apply the method of the preceding chapter to another question which is not directly connected with the subject of partitions. In the binomial and multinomial expansions, the exponent being a positive integer, every coefficient is an integer. This fact depends analytically upon the circumstance that the product of any m consecutive integers is divisible by factorial m; we have

$$\frac{n+1}{1} \cdot \frac{n+2}{2} \dots \frac{n+m}{m}$$

an integer for all values of n.

The present question is the determination of all values of $\alpha_1, \alpha_2, \dots \alpha_m$ for which the expression

$$\left(\frac{n+1}{1}\right)^{\alpha_1} \left(\frac{n+2}{2}\right)^{\alpha_2} \dots \left(\frac{n+m}{m}\right)^{\alpha_m}$$

is an integer for all values of n. In particular the discovery of the finite number of fundamental products of this nature, for a given value of m, from which all the forms may be generated by multiplication; and the complete syzygetic theory.

We write the expression in the notation

$$N_1^{\alpha_1} N_2^{\alpha_2} \dots N_m^{\alpha_m},$$

and observe that it is derived from the product

$$X_1^{\alpha_1} X_2^{\alpha_2} \dots X_m^{\alpha_m}$$

discussed in the previous chapter by putting x equal to unity. All the algebraic products which are finite and integral give rise in this manner to arithmetical products which are integers. This however is as much as can be said, for beyond this the two theories proceed upon widely divergent lines. As might be expected, the arithmetical products form a more extended system than the algebraical.

400. To commence with the Order Two or

$$\left(\frac{n+1}{1}\right)^{\alpha_1}\left(\frac{n+2}{2}\right)^{\alpha_2} = N_1^{\alpha_1} N_2^{\alpha_2},$$

the product is an integer when n is even, but when n is uneven we must have

$$\alpha_1 \geqslant \alpha_2.$$

Hence
$$\Sigma N_1^{\alpha_1} N_2^{\alpha_2} = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda N_1)\left(1-\frac{1}{\lambda} N_2\right)}$$
$$= \frac{1}{(1-N_1)(1-N_1N_2)},$$

shewing the trivial result that N_1 and N_1N_2 are fundamental forms.

401. For the Order Three, where

$$\left(\frac{n+1}{1}\right)^{\alpha_1}\left(\frac{n+2}{2}\right)^{\alpha_2}\left(\frac{n+3}{3}\right)^{\alpha_3} = N_1^{\alpha_1} N_2^{\alpha_2} N_3^{\alpha_3},$$

we have to consider the divisibility of the numerator numbers by powers of 2 and 3, and since if n be uneven one of the numbers $n+1$, $n+3$ must be divisible by 4 it is necessary to investigate the conditions according as n is of the forms $4m+1$, $4m+3$, $3m+1$, $3m+2$, the forms $4m$, $4m+2$, $3m$ yielding no conditions.

When n is of the form	condition is	auxiliary
$4m+1$	$\alpha_1 + 2\alpha_3 \geqslant \alpha_2$	λ_1
$4m+3$	$2\alpha_1 + \alpha_3 \geqslant \alpha_2$	
$3m+1$	$\alpha_2 \geqslant \alpha_3$	λ_2
$3m+2$	$\alpha_1 \geqslant \alpha_3$	λ_3

It will be noted that these inequalities are not independent since addition of the first and fourth produces the second. We therefore omit the second as superfluous and assign auxiliaries λ_1, λ_2, λ_3 as shewn. We have therefore

$$\Sigma N_1^{\alpha_1} N_2^{\alpha_2} N_3^{\alpha_3} = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_3 N_1)\left(1-\frac{\lambda_2}{\lambda_1} N_2\right)\left(1-\frac{\lambda_1^2}{\lambda_2\lambda_3} N_3\right)}.$$

Eliminating λ_2 this is

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_3 N_1)\left(1-\frac{1}{\lambda_1} N_2\right)\left(1-\frac{\lambda_1}{\lambda_3} N_2 N_3\right)},$$

and eliminating λ_3

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 N_1)\left(1-\frac{1}{\lambda_1} N_2\right)(1-\lambda_1^2 N_1 N_2 N_3)},$$

and this by a formula on a previous page is

$$\frac{1 - N_1^2N_2^2N_3 - N_1^2N_2^3N_3 - N_1^2N_2^4N_3^2 + N_1^3N_2^4N_3^2 + N_1^3N_2^5N_3^2}{(1-N_1)(1-N_1N_2)(1-N_1N_2N_3)(1-N_1N_2^2N_3)(1-N_1N_2^3N_3)},$$

establishing the new fundamental products

$$N_1N_2N_3, \quad N_1N_2^2N_3, \quad N_1N_2^3N_3,$$

connected by syzygies as shewn by the numerator terms.

We observe one form $N_1N_2^3N_3$ which is not derivable from the algebraic theory of the previous chapter.

402. The forms are exponentially symmetrical. On proceeding to the Fourth Order unsymmetrical forms appear for the first time. A few experiments shew that the unsymmetrical forms are of the form

$$N_1^{\alpha_1}N_2^{\alpha_2}N_4^{\alpha_4},$$

so that we may divide the investigation into two cases.

Case I. The form $\qquad N_1^{\alpha_1}N_2^{\alpha_2}N_3^{\alpha_3}N_4^{\alpha_4}.$

Case II. The form $\qquad N_1^{\alpha_1}N_2^{\alpha_2}N_4^{\alpha_4}.$

The conditions for the general form $N_1^{\alpha_1}N_2^{\alpha_2}N_3^{\alpha_3}N_4^{\alpha_4}$, obtained by discussing n in the forms $4m+1$, $4m+2$, $4m+3$, $3m+1$, $3m+2$, are

$$\begin{gathered}\alpha_1+2\alpha_3 \geqslant \alpha_2+2\alpha_4,\\ \alpha_2 \geqslant \alpha_4,\\ 2\alpha_1+\alpha_3 \geqslant \alpha_2+2\alpha_4,\\ \alpha_2 \geqslant \alpha_3,\\ \alpha_1+\alpha_4 \geqslant \alpha_3.\end{gathered}$$

For *Case I* these become

$$\begin{gathered}\alpha_2 \geqslant \alpha_1,\\ 2\alpha_1 \geqslant \alpha_2,\end{gathered}$$

leading to

$$\underset{\geqslant}{\Omega}\frac{1}{\left(1-\frac{\lambda_2^2}{\lambda_1}N_1N_4\right)\left(1-\frac{\lambda_1}{\lambda_2}N_2N_3\right)}$$

$$=\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_2N_1N_2N_3N_4)\left(1-\frac{1}{\lambda_2}N_2N_3\right)}$$

$$=\frac{1}{(1-N_1N_2N_3N_4)(1-N_1N_2^2N_3^2N_4)},$$

and we have merely the fundamental forms derivable from the algebraic theory.

For *Case II* the conditions become

$$\begin{gathered}\alpha_1 \geqslant \alpha_2+2\alpha_4,\\ \alpha_2 \geqslant \alpha_4,\end{gathered}$$

leading to

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_1 N_1)\left(1-\frac{\lambda_2}{\lambda_1}N_2\right)\left(1-\frac{1}{\lambda_1^2\lambda_2}N_4\right)}$$

$$=\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_1 N_1)\left(1-\frac{1}{\lambda_1}N_2\right)\left(1-\frac{1}{\lambda_1^3}N_2N_4\right)}$$

$$=\frac{1}{(1-N_1)(1-N_1N_2)(1-N_1^3N_2N_4)},$$

establishing the new fundamental form

$$N_1^3N_2N_4.$$

Forms of this species occur in the case of every even order but never present themselves for an uneven order.

The general form is

$$N_1^{\alpha_1}N_2^{\alpha_2}N_4^{\alpha_4}N_6^{\alpha_6}\ldots N_{2s}^{\alpha_{2s}}.$$

403. For the case of the Order Five there is a very complicated system of fundamental forms. They appear to be, by an investigation which is too long to reproduce here, thirteen in number, viz.

$$N_1N_2N_3N_4N_5,$$
$$N_1N_2N_3^2N_4N_5,$$
$$N_1N_2^2N_3^2N_4^2N_5,$$
$$N_1N_2^2N_3^3N_4^2N_5,$$

four which are derivable from the previous algebraic theory;

$$N_1^2N_2^5N_3^7N_4^5N_5^2,$$
$$N_1^3N_2^4N_3^3N_4^4N_5^3,$$

two which are symmetrical but do not appertain to the algebraic theory;

$$N_1N_2^2N_3N_4N_5,$$
$$N_1N_2^2N_3^2N_4N_5,$$
$$N_1N_2^3N_3^2N_4N_5,$$
$$N_1N_2^3N_3^3N_4^2N_5,$$
$$N_1N_2^4N_3^2N_4N_5,$$
$$N_1^3N_2^2N_3N_4^2N_5,$$
$$N_1^3N_2^3N_3^2N_4^3N_5^2,$$

and seven which are unsymmetrical.

It will be useful to verify these forms *à posteriori*.

With the first four we need not deal.

As regards the next two which are symmetrical we note that the conditions that $N_1^{\alpha_1}N_2^{\alpha_2}N_3^{\alpha_3}N_4^{\alpha_4}N_5^{\alpha_5}$ may be an integer, obtainable by discussing n in the

forms $4m+1$, $4m+2$, $4m+3$, $3m+1$, $3m+2$, $5m+1$, $5m+2$, $5m+3$, $5m+4$ are

$$\begin{aligned} \alpha_1 + 2\alpha_3 + \alpha_5 &\geqslant \alpha_2 + 2\alpha_4, \\ \alpha_2 &\geqslant \alpha_4, \\ 2\alpha_1 + \alpha_3 + 2\alpha_5 &\geqslant \alpha_2 + 2\alpha_4, \\ \alpha_2 + \alpha_5 &\geqslant \alpha_3, \\ \alpha_1 + \alpha_4 &\geqslant \alpha_3, \\ \alpha_4 &\geqslant \alpha_5, \\ \alpha_3 &\geqslant \alpha_5, \\ \alpha_1 &\geqslant \alpha_5, \end{aligned}$$

that corresponding to $5m+3$ being implied by the remainder.

Putting herein $\alpha_5 = \alpha_1$, $\alpha_4 = \alpha_2$ and further $\alpha_3 = \alpha_1 + \alpha_2$ in order to find the form $N_1^2 N_2^5 N_3^7 N_4^5 N_5^2$ and to exclude some of the others, we find

$$4\alpha_1 \geqslant \alpha_2, \quad 5\alpha_1 \geqslant 2\alpha_2, \quad \alpha_2 \geqslant \alpha_1$$

for the form

$$(N_1 N_5)^{\alpha_1} (N_2 N_4)^{\alpha_2} N_3^{\alpha_1+\alpha_2} \equiv (N_1 N_3 N_5)^{\alpha_1} (N_2 N_3 N_4)^{\alpha_2},$$

and we are led to

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda_1^4 \lambda_2^5}{\lambda_3} N_1 N_3 N_5\right)\left(1 - \dfrac{\lambda_3}{\lambda_1 \lambda_2^2} N_2 N_3 N_4\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda_1^3 \lambda_2^3 N_1 N_2 N_3^2 N_4 N_5)\left(1 - \dfrac{1}{\lambda_1 \lambda_2^2} N_2 N_3 N_4\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1 + \lambda_2 N_1 N_2^2 N_3^3 N_4^2 N_5 + \dfrac{1}{\lambda_2} N_1 N_2^3 N_3^4 N_4^3 N_5}{(1 - \lambda_2^3 N_1 N_2 N_3^2 N_4 N_5)\left(1 - \dfrac{1}{\lambda_2^3} N_1 N_2^4 N_3^5 N_4^4 N_5\right)}$$

$$= \frac{1 + N_1 N_2^2 N_3^3 N_4^2 N_5 + N_1^2 N_2^4 N_3^6 N_4^4 N_5^2}{(1 - N_1 N_2 N_3^2 N_4 N_5)(1 - N_1^2 N_2^5 N_3^7 N_4^5 N_5^2)},$$

establishing the form $N_1^2 N_2^5 N_3^7 N_4^5 N_5^2$.

If we take the special condition $\alpha_3 = \alpha_1$ instead of $\alpha_3 = \alpha_1 + \alpha_2$ the conditions become

$$4\alpha_1 \geqslant 3\alpha_2, \quad \alpha_2 \geqslant \alpha_1,$$

and we are led to

$$\underset{\geqslant}{\Omega} \frac{1}{\left(1 - \dfrac{\lambda_1^4}{\lambda_2} N_1 N_3 N_5\right)\left(1 - \dfrac{\lambda_2}{\lambda_1^3} N_2 N_4\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1 - \lambda_1 N_1 N_2 N_3 N_4 N_5)\left(1 - \dfrac{1}{\lambda_1^3} N_2 N_4\right)}$$

$$= \frac{1}{(1 - N_1 N_2 N_3 N_4 N_5)(1 - N_1^3 N_2^4 N_3^3 N_4^4 N_5^3)},$$

establishing the form $N_1^3 N_2^4 N_3^3 N_4^4 N_5^3$.

For the unsymmetrical forms, it will be observed that as regards the first five of them $\alpha_5 = \alpha_1$, $\alpha_2 > \alpha_4$; and in the last two $\alpha_1 > \alpha_5$, $\alpha_2 = \alpha_4$. Making use of these special conditions the forms are readily established.

For the Order Six we will examine the form

$$N_1^{\alpha_1} N_2^{\alpha_2} N_4^{\alpha_4} N_6^{\alpha_6}$$
$$\equiv \left(\frac{n+1}{1}\right)^{\alpha_1} \left(\frac{n+2}{2}\right)^{\alpha_2} \left(\frac{n+4}{4}\right)^{\alpha_4} \left(\frac{n+6}{6}\right)^{\alpha_6}.$$

The conditions are easily seen to be

$$\begin{aligned} \alpha_1 &\geqslant \alpha_2 + 2\alpha_4 + \alpha_6, \\ \alpha_2 + \alpha_6 &\geqslant \alpha_4, \\ \alpha_2 &\geqslant \alpha_6, \\ \alpha_1 + \alpha_4 &\geqslant \alpha_6, \end{aligned}$$

n being of the forms $4m+1$, $4m+2$, $3m+1$, $3m+2$ respectively.

Obviously the fourth condition is superfluous.

We find

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1 N_1)\left(1-\frac{\lambda_2\lambda_3}{\lambda_1} N_2\right)\left(1-\frac{1}{\lambda_1^2\lambda_2} N_4\right)\left(1-\frac{\lambda_2}{\lambda_1\lambda_3} N_6\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1-N_1)(1-\lambda_2\lambda_3 N_1 N_2)\left(1-\frac{1}{\lambda_2} N_1^2 N_4\right)\left(1-\frac{\lambda_2}{\lambda_3} N_1 N_6\right)}$$

$$= \underset{\geqslant}{\Omega} \frac{1}{(1-N_1)(1-\lambda_2 N_1 N_2)\left(1-\frac{1}{\lambda_2} N_1^2 N_4\right)(1-\lambda_2^2 N_1^2 N_2 N_6)}$$

$$= \frac{1 - N_1^5 N_2^2 N_4 N_6 - N_1^7 N_2^2 N_4^2 N_6 - N_1^8 N_2^2 N_4^2 N_6^2 + N_1^9 N_2^3 N_4^2 N_6^2 + N_1^{11} N_2^3 N_4^3 N_6^2}{(1-N_1)(1-N_1^2 N_2 N_6)(1-N_1 N_2)(1-N_1^3 N_2 N_4)(1-N_1^6 N_2 N_4^2 N_6)(1-N_1^4 N_2 N_4 N_6)},$$

establishing the fundamental forms

$$N_1^2 N_2 N_6,$$
$$N_1^4 N_2 N_4 N_6,$$
$$N_1^6 N_2 N_4^2 N_6.$$

Similarly for the Order Eight it is easy to find the fundamental forms

$$N_1^7 N_2 N_4 N_6 N_8, \quad N_1^8 N_2 N_4 N_6^2 N_8, \quad N_1^8 N_2^3 N_4 N_8,$$
$$N_1^{10} N_2 N_4^2 N_6^2 N_8, \quad N_1^{22} N_2^2 N_4^3 N_6^5 N_8^3.$$

CHAPTER VII

THE THEORY OF MAGIC SQUARES

404. The method of Partition Analysis is applicable to all arrangements of integers which are defined by homogeneous linear Diophantine Equalities or Inequalities; and when the properties which result from the definition persist after addition of corresponding numbers a syzygetic theory results. Many instances of configurations of integers *in plano* or *in solido* will occur to the mind as having been subjects of contemplation by mathematicians and others from the earliest times. Amongst these are those of the nature of the magic square. The properties of such squares are usually stated in terms of the sums of the numbers which appear in the rows, columns and diagonals.

In an ordinary magic square of order n, the square is divided up into n^2 compartments; the n^2 compartment numbers are all different, being usually the first n^2 natural numbers, and the arrangement is such that every row, every column and each diagonal contain numbers which have the same sum.

As distinct from an ordinary magic square a "general magic square" is defined to consist of n^2 integers arranged in the compartments in such wise that the numbers in the rows, columns and diagonals add to the same number, zeros and repetitions of the same number being permissible among the integers.

We may regard general magic squares of the same order n as numerical magnitudes. To add two such magnitudes we add together the numbers in corresponding positions or compartments to form a new magnitude which obviously is also a "general magic square." We can form a linear function of such magnitudes, of the same order, the coefficients being positive integers, and it also will be a general magic square. An ordinary magic square is a particular case of a general magic square, but the particular property of an ordinary magic square that it must involve different integers in the compartments prevents the formation of new squares by addition, so that within the group of such squares there is no syzygetic theory.

405. We proceed to investigate the syzygetic theory of general magic squares of the Third Order. Such squares do not exist in the case of the Second Order except in the trivial case

$$\begin{matrix} a & a \\ a & a \end{matrix}$$

but there are squares in which the row and column but *not* the diagonal properties are in evidence.

For let such a square be

$$\begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{matrix}$$

This must clearly have the form

$$\begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{matrix}$$

where if α_5 be some integer

$$\alpha_1 + \alpha_2 = \alpha_5,$$

and we find

$$\Sigma x_1^{\alpha_1} x_2^{\alpha_2} x_5^{\alpha_5} = \underset{=}{\Omega} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)\left(1-\frac{1}{\lambda}x_5\right)},$$

wherein we recall that the symbol $\underset{=}{\Omega}$ denotes that in the ascending expansion of the function to which it is prefixed we are to retain only those terms which are free from the auxiliary λ.

The expression clearly has the value

$$\frac{1}{(1-x_1x_5)(1-x_2x_5)},$$

the denominator factors denoting the fundamental solutions

α_1	α_2	α_5
1	0	1
0	1	1

Thus we have the fundamental squares

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

and the most general square of the type

$$\alpha_1 \left|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right. + \alpha_2 \left|\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right. = \left|\begin{matrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{matrix}\right.$$

a trivial result.

406. Passing to the third order let the square be

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{matrix}$$

and let the number associated alike with the sum of each row, each and each diagonal be α_{10}; we then have the eight equalities

$$\alpha_1+\alpha_2+\alpha_3=\alpha_4+\alpha_5+\alpha_6=\alpha_7+\alpha_8+\alpha_9=\alpha_{10},$$
$$\alpha_1+\alpha_4+\alpha_7=\alpha_2+\alpha_5+\alpha_8=\alpha_3+\alpha_6+\alpha_9=\alpha_{10},$$
$$\alpha_1+\alpha_5+\alpha_9=\alpha_3+\alpha_5+\alpha_7=\alpha_{10}.$$

We require the sum

$$\Sigma x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}x_5^{\alpha_5}x_6^{\alpha_6}x_7^{\alpha_7}x_8^{\alpha_8}x_9^{\alpha_9}x_{10}^{\alpha_{10}}$$

for all sets of integers satisfying the equalities.

Associate with the equalities the auxiliaries

$$\begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \end{matrix}$$

respectively, each equality having α_{10} on the dexter, so that λ_2 is associated with $\alpha_4+\alpha_5+\alpha_6=\alpha_{10}$.

The sum is

$$\underset{=}{\Omega}\frac{1}{\left\{\begin{matrix}(1-\lambda_1\lambda_4\lambda_7x_1)(1-\lambda_1\lambda_5x_2)(1-\lambda_1\lambda_6\lambda_8x_3)(1-\lambda_2\lambda_4x_4)(1-\lambda_2\lambda_5\lambda_7\lambda_8x_5)\\(1-\lambda_2\lambda_6x_6)(1-\lambda_3\lambda_4\lambda_8x_7)(1-\lambda_3\lambda_5x_8)(1-\lambda_3\lambda_6\lambda_7x_9)\left(1-\dfrac{1}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8}x_{10}\right)\end{matrix}\right\}}.$$

Eliminating $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ in any convenient order, this is found to have the expression

$$\underset{=}{\Omega}\frac{1}{\left\{\begin{matrix}(1-\lambda_7^2x_1x_5x_9x_{10})\left(1-\dfrac{1}{\lambda_7}x_2x_6x_7x_{10}\right)\left(1-\dfrac{1}{\lambda_7}x_3x_4x_8x_{10}\right)\\(1-\lambda_8^2x_3x_5x_7x_{10})\left(1-\dfrac{1}{\lambda_8}x_2x_4x_9x_{10}\right)\end{matrix}\right\}}$$

$$+\underset{=}{\Omega}\frac{\dfrac{1}{\lambda_8^2}x_1x_2x_4x_6x_8x_9x_{10}^2}{\left\{\begin{matrix}(1-\lambda_7^2x_1x_5x_9x_{10})\left(1-\dfrac{1}{\lambda_7}x_2x_6x_7x_{10}\right)\left(1-\dfrac{1}{\lambda_7}x_3x_4x_8x_{10}\right)\\\left(1-\dfrac{1}{\lambda_8}x_2x_4x_9x_{10}\right)\left(1-\dfrac{1}{\lambda_8^2}x_1x_2x_4x_6x_8x_9x_{10}^2\right)\end{matrix}\right\}}$$

$$+\underset{=}{\Omega}\frac{\dfrac{1}{\lambda_8}x_1x_6x_8x_{10}(1-x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3)}{\left\{\begin{matrix}(1-\lambda_7^2x_1x_5x_9x_{10})\left(1-\dfrac{1}{\lambda_7}x_2x_6x_7x_{10}\right)\left(1-\dfrac{1}{\lambda_7}x_3x_4x_8x_{10}\right)\\(1-\lambda_8^2x_3x_5x_7x_{10})\left(1-\dfrac{1}{\lambda_8}x_1x_6x_8x_{10}\right)\left(1-\dfrac{1}{\lambda_8^2}x_1x_2x_4x_6x_8x_9x_{10}^2\right)\end{matrix}\right\}},$$

and we may pause to draw some deductions.

407. If the general magic squares we are considering are deprived of the properties of the two diagonals we have merely now to put $\lambda_7 = \lambda_8 = 1$.

The real generating function then reduces to

$$\frac{1 - x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3}{\left\{\begin{matrix}(1 - x_1x_5x_9x_{10})(1 - x_1x_6x_8x_{10})(1 - x_2x_4x_9x_{10}) \\ (1 - x_2x_6x_7x_{10})(1 - x_3x_4x_8x_{10})(1 - x_3x_5x_7x_{10})\end{matrix}\right\}},$$

indicating the fundamental products

$$\begin{matrix} x_1x_5x_9x_{10} & x_1x_6x_8x_{10} & x_2x_4x_9x_{10} \\ x_2x_6x_7x_{10} & x_3x_4x_8x_{10} & x_3x_5x_7x_{10} \end{matrix}$$

connected by the simple syzygy

$$x_1x_5x_9x_{10} \,.\, x_2x_6x_7x_{10} \,.\, x_3x_4x_8x_{10} - x_1x_6x_8x_{10} \,.\, x_2x_4x_9x_{10} \,.\, x_3x_5x_7x_{10} = 0,$$

corresponding to the fundamental squares

$$\begin{matrix} x_1x_5x_9x_{10} & & x_2x_6x_7x_{10} & & x_3x_4x_8x_{10} \\ \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & & \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & & \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix} \\ \\ x_1x_6x_8x_{10} & & x_2x_4x_9x_{10} & & x_3x_5x_7x_{10} \\ \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} & & \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} & & \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix} \end{matrix}$$

(the exponent of x_{10} being unity shewing that the number associated with the rows and columns is unity)

connected by the simple syzygy

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} + \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} + \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix}$$

$$= \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} + \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} + \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}$$

each side being equal to

$$\begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix} \equiv x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3.$$

This is the complete syzygetic theory of these squares.

To obtain the enumerating generating function we put

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = x \text{ and } x_{10} = y,$$

when we find

$$\frac{1 - x^9 y^3}{(1 - x^3 y)^6}.$$

In this the coefficient of $x^{3n} y^n$ represents the number of squares such that in each column and in each row the sum of the numbers is n. It has the value $3\binom{n+3}{4} + \binom{n+2}{2}$.

408. Resuming the discussion of Art. 406 we eliminate λ_7 and find

$$\underset{=}{\Omega} \frac{1 + x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}^3}{\left\{\begin{array}{c}(1 - x_1 x_3^2 x_4^2 x_5 x_8^2 x_9 x_{10}^3)(1 - x_1 x_2^2 x_5 x_6^2 x_7^2 x_9 x_{10}^3) \\ (1 - \lambda_8^2 x_3 x_5 x_7 x_{10})\left(1 - \frac{1}{\lambda_8} x_2 x_4 x_9 x_{10}\right)\end{array}\right\}}$$

$$+ \underset{=}{\Omega} \frac{\frac{1}{\lambda_8} x_1 x_6 x_8 x_{10} (1 - x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 x_7^2 x_8^2 x_9^2 x_{10}^6)}{\left\{\begin{array}{c}(1 - x_1 x_3^2 x_4^2 x_5 x_8^2 x_9 x_{10}^3)(1 - x_1 x_2^2 x_5 x_6^2 x_7^2 x_9 x_{10}^3) \\ (1 - \lambda_8^2 x_3 x_5 x_7 x_{10})\left(1 - \frac{1}{\lambda_8} x_1 x_6 x_8 x_{10}\right)\left(1 - \frac{1}{\lambda_8^2} x_1 x_2 x_4 x_6 x_8 x_9 x_{10}^2\right)\end{array}\right\}},$$

and the reader should observe that the second fraction of Art. 406 has the value zero and contributes nothing to this result. But if we wish to discuss the squares which have the principal diagonal property but not that associated with the secondary diagonal, we do not omit the second fraction referred to but add it to the two just obtained and then put $\lambda_8 = 1$. We then get on simplification

$$\frac{1 - x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 x_7^2 x_8^2 x_9^2 x_{10}^6}{\left\{\begin{array}{c}(1 - x_1 x_6 x_8 x_{10})(1 - x_2 x_4 x_9 x_{10})(1 - x_3 x_5 x_7 x_{10}) \\ (1 - x_1 x_2^2 x_5 x_6^2 x_7^2 x_9 x_{10}^3)(1 - x_1 x_3^2 x_4^2 x_5 x_8^2 x_9 x_{10}^3)\end{array}\right\}},$$

establishing the five fundamental products

$$x_1 x_6 x_8 x_{10}, \quad x_2 x_4 x_9 x_{10}, \quad x_3 x_5 x_7 x_{10},$$
$$x_1 x_2^2 x_5 x_6^2 x_7^2 x_9 x_{10}^3, \quad x_1 x_3^2 x_4^2 x_5 x_8^2 x_9 x_{10}^3,$$

connected by the simple syzygy

$$(x_1 x_6 x_8 x_{10})^2 (x_2 x_4 x_9 x_{10})^2 (x_3 x_5 x_7 x_{10})^2 = (x_1 x_2^2 x_5 x_6^2 x_7^2 x_9 x_{10}^3)(x_1 x_3^2 x_4^2 x_5 x_8^2 x_9 x_{10}^3),$$

corresponding to the fundamental squares

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \qquad \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}$$

$$\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{array}$$

connected by the simple syzygy

$$2\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}+2\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix}+2\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}$$

$$=\begin{vmatrix}1&2&0\\0&1&2\\2&0&1\end{vmatrix}+\begin{vmatrix}1&0&2\\2&1&0\\0&2&1\end{vmatrix}$$

squares, all of them, such that the principal diagonal is associated with the same number as the rows and columns.

Similarly if it is the secondary diagonal that is to be associated with the rows and columns we arrive at the real generating function

$$\frac{1-x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2x_7^2x_8^2x_9^2x_{10}^6}{\left\{\begin{matrix}(1-x_1x_5x_9x_{10})(1-x_2x_6x_7x_{10})(1-x_3x_4x_8x_{10})\\(1-x_1^2x_3x_5x_6^2x_7x_8^2x_{10}^3)(1-x_2^2x_3x_4^2x_5x_7x_9^2x_{10}^3)\end{matrix}\right\}},$$

from which the fundamental squares and the syzygy connecting them can be written down.

In both cases of the *one diagonal* theory the enumerating function is

$$\frac{1-x^{18}y^6}{(1-x^3y)^3(1-x^9y^3)^2},$$

in which the coefficient of $x^{3n}y^n$ must be sought in order to enumerate the squares such that each row, each column and one selected diagonal are to be associated with the number n.

409. Resuming the discussion we finally eliminate λ_8 and arrive at the real generating function

$$\frac{1-2x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2x_7^2x_8^2x_9^2x_{10}^6+x_1^4x_2^4x_3^4x_4^4x_5^4x_6^4x_7^4x_8^4x_9^4x_{10}^{12}}{\left\{\begin{matrix}(1-x_1x_2^2x_5x_6^2x_7^2x_9x_{10}^3)(1-x_1x_3^2x_4^2x_5x_8^2x_9x_{10}^3)(1-x_1^2x_3x_5x_6^2x_7x_8^2x_{10}^3)\\(1-x_2^2x_3x_4^2x_5x_7x_9^2x_{10}^3)(1-x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3)\end{matrix}\right\}},$$

indicating the fundamental products

$$x_1x_2^2x_5x_6^2x_7^2x_9x_{10}^3,$$
$$x_1x_3^2x_4^2x_5x_8^2x_9x_{10}^3,$$
$$x_1^2x_3x_5x_6^2x_7x_8^2x_{10}^3,$$
$$x_2^2x_3x_4^2x_5x_7x_9^2x_{10}^3,$$
$$x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3,$$

connected by the first syzygies

$$x_1x_2^2x_5x_6^2x_7^2x_9x_{10}^3\,.\,x_1x_3^2x_4^2x_5x_8^2x_9x_{10}^3$$
$$=x_1^2x_3x_5x_6^2x_7x_8^2x_{10}^3\,.\,x_2^2x_3x_4^2x_5x_7x_9^2x_{10}^3$$
$$=(x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}^3)^2,$$

and writing these

$$AB = CD = E^2,$$

we have the second order syzygy

$$E^2(AB - E^2) + AB(CD - E^2) = 0,$$

corresponding to the fundamental squares

$$\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{matrix} \qquad \begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix} \qquad \begin{matrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

in which *both diagonals* share in the property of the rows and columns.

The first syzygies connecting them may be written

$$\left|\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix}\right. + \left|\begin{matrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{matrix}\right. = 2\left|\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}\right. = \left|\begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix}\right. + \left|\begin{matrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{matrix}\right.$$

The enumerating function is

$$\frac{(1 - x^{18}y^6)^2}{(1 - x^9y^3)^5},$$

in which the coefficient of $x^{3n}y^n$ must be sought.

Moreover n must be of the form $3m$ so that we have to seek the coefficient of $x^{3m}y^m$ in

$$(1 - x^6y^2)^2(1 - x^3y)^{-5},$$

and this is in fact $m^2 + (m+1)^2$.

The ordinary magic squares, the component integers being 1, 2, 3, 4, 5, 6, 7, 8, 9, are eight in number. Calling the fundamental squares as written above

$$\begin{matrix} A & B & C & D \\ & E & & \end{matrix}$$

these are

$$A + 3C + E = \left|\begin{matrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{matrix}\right., \quad 3A + C + E = \left|\begin{matrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{matrix}\right.$$

$$A + 3D + E = \left|\begin{matrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{matrix}\right., \quad 3A + D + E = \left|\begin{matrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{matrix}\right.$$

$$B+3C+E=\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}, \quad 3B+C+E=\begin{vmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{vmatrix}$$

$$B+3D+E=\begin{vmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{vmatrix}, \quad 3B+D+E=\begin{vmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{vmatrix}$$

There is no theoretical difficulty in dealing with the squares of higher orders, but even in the case $n=4$ there is practical difficulty in handling the $\underset{=}{\Omega}$ generating function.

410. Apart from the syzygetic theory the enumeration of general magic squares of given order and given number associated with the rows, columns and diagonals may be viewed in the following manner.

If we are not concerned with diagonal properties the squares have already been enumerated in Section V.

If h_m denotes the sum of the homogeneous products of degree m of the n magnitudes

$$\alpha_1, \alpha_2, \ldots \alpha_n,$$

and h_m be raised to the power n and be developed so as to be a linear function of monomial symmetric functions, the coefficient of

$$\alpha_1^m \alpha_2^m \ldots \alpha_n^m$$

is the number of squares of order n which have the property that the sum of the numbers in each row and in each column is m. (See Section V, Art. 199.) Thus if $m=3$, $n=2$,

$$h_3^2=(\alpha_1^3+\alpha_2^3+\alpha_1^2\alpha_2+\alpha_1\alpha_2^2)^2=\ldots+4\alpha_1^3\alpha_2^3+\ldots,$$

the coefficient 4 enumerating the four squares

$$\begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix} \qquad \begin{matrix} 0 & 3 \\ 3 & 0 \end{matrix} \qquad \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

411. To introduce the diagonal properties, let $h_{m,s}$ denote h_m when in it α_s, α_{n-s+1} are replaced by $\lambda\alpha_s$, $\mu\alpha_{n-s+1}$ respectively.

Form the product

$$h_{m,1}\, h_{m,2} \ldots h_{m,n},$$

and it will be found that the coefficient herein, after development, of

$$\lambda^m \mu^m \alpha_1^m \alpha_2^m \ldots \alpha_n^m,$$

is the number of general magic squares of order n, associated with the number m

To see how this arises take $n=4$, $m=1$ and form a product

$$\begin{aligned}&(\lambda\alpha_1 + \alpha_2 + \alpha_3 + \mu\alpha_4)\\ \times &(\alpha_1 + \lambda\alpha_2 + \mu\alpha_3 + \alpha_4)\\ \times &(\alpha_1 + \mu\alpha_2 + \lambda\alpha_3 + \alpha_4)\\ \times &(\mu\alpha_1 + \alpha_2 + \alpha_3 + \lambda\alpha_4).\end{aligned}$$

When we multiply out in order to obtain the term involving

$$\lambda\mu\alpha_1\alpha_2\alpha_3\alpha_4,$$

we must take one portion of the term from each of the four factors and regarding the elements of the four factors as corresponding to the compartments of a square of order 4, this means that in picking out the product $\lambda\mu\alpha_1\alpha_2\alpha_3\alpha_4$ we take an element from each row of the square. Again since α_1, α_2, α_3, α_4 only occur in the first, second, third and fourth columns of the square respectively we must also take an element from each column of the square. Finally in order to obtain $\lambda\mu$ we must choose elements from each diagonal of the square.

Ex. gr. If we were to select in the multiplication α_2, $\mu\alpha_3$, α_1, $\lambda\alpha_4$ from the first, second, third and fourth factors respectively, the selection would correspond to the square

$$\begin{vmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}$$

412. Again take $n=4$, $m=2$. We form the product

$$\begin{aligned}&\{\lambda^2\alpha_1^2 + \lambda\mu\alpha_1\alpha_4 + \mu^2\alpha_4^2 + (\lambda\alpha_1 + \mu\alpha_4)(\alpha_2 + \alpha_3) + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2\}\\ \times &\{\lambda^2\alpha_2^2 + \lambda\mu\alpha_2\alpha_3 + \mu^2\alpha_3^2 + (\lambda\alpha_2 + \mu\alpha_3)(\alpha_1 + \alpha_4) + \alpha_1^2 + \alpha_1\alpha_4 + \alpha_4^2\}\\ \times &\{\lambda^2\alpha_3^2 + \lambda\mu\alpha_2\alpha_3 + \mu^2\alpha_2^2 + (\lambda\alpha_3 + \mu\alpha_2)(\alpha_1 + \alpha_4) + \alpha_1^2 + \alpha_1\alpha_4 + \alpha_4^2\}\\ \times &\{\lambda^2\alpha_4^2 + \lambda\mu\alpha_1\alpha_4 + \mu^2\alpha_1^2 + (\lambda\alpha_4 + \mu\alpha_1)(\alpha_2 + \alpha_3) + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2\}.\end{aligned}$$

In forming the term which involves the product

$$\lambda^2\mu^2\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2,$$

we regard the successive factors as corresponding to the successive rows of the square; *the suffix of the* α as denoting the column, and λ, μ as corresponding to the diagonals. The exponent of the α represents the number which is to be entered in the compartment of the square, this compartment being specified by the suffix of the α and the factor in which the α occurs.

Thus if in performing the multiplication we select the terms

$$\lambda^2\alpha_1^2,\quad \mu\alpha_3\alpha_4,\quad \mu\alpha_2\alpha_4,\quad \alpha_2\alpha_3,$$

from the four factors respectively, we have the corresponding square

$$\begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

In whatever manner we select the factor elements we have in correspondence a general magic square.

These examples suffice to establish that the number which enumerates the magic squares is identical with the coefficient of

$$\lambda^m \mu^m (\alpha_1 \alpha_2 \ldots \alpha_n)^m,$$

which arises from the product

$$h_{m,1} h_{m,2} \ldots h_{m,n}.$$

If we wish to make any restriction in regard to the numbers that appear in the sth row of the square we have merely to strike out certain terms from the function

$$h_{m,s}.$$

For example if no number is to exceed t we have merely to strike out all terms which involve exponents which exceed t.

If the sth row is to involve numbers which collectively are included in certain specified partitions of m we merely strike out from $h_{m,s}$ all terms whose exponents do not collectively correspond to one of such partitions.

If the sth column of the square is not to include the number t we must in each factor strike out every term which involves α_s^t. In fact each and all of the n^2 numbers in the square can be restricted in any desired manner.

If ordinary magic squares are under consideration we must from $h_{m,s}$ (for all values of s), m having now a fixed value in relation to n, strike out all terms which involve a repetition of exponent. Also in picking out the n elements which form $\lambda^m \mu^m \alpha_1^m \alpha_2^m \ldots \alpha_n^m$ we must take care that α_s (for all values of s) never appears twice with the same exponent. In both cases zero counts as an exponent. Also neither λ nor μ in the elements must occur twice with the same exponent, zero being counted as an exponent.

413. To resume the general subject we will now consider the enumeration of the squares which are such that the sum of the numbers in each row and in each column is unity, the sums of the numbers in the λ and μ diagonals being any given numbers.

Observe first that if the numbers occurring in the diagonals are immaterial the number of the squares is $n\,!$ because

$$h^n = \ldots + n\,!\, \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n.$$

Consider an even order $2n$ and form the product

$$\begin{aligned}
&(\lambda\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{2n-2} + \alpha_{2n-1} + \mu\alpha_{2n})\\
\times&(\alpha_1 + \lambda\alpha_2 + \alpha_3 + \dots + \alpha_{2n-2} + \mu\alpha_{2n-1} + \alpha_{2n})\\
\times&(\alpha_1 + \alpha_2 + \lambda\alpha_3 + \dots + \mu\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n})\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\times&(\alpha_1 + \alpha_2 + \mu\alpha_3 + \dots + \lambda\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n})\\
\times&(\alpha_1 + \mu\alpha_2 + \alpha_3 + \dots + \alpha_{2n-2} + \lambda\alpha_{2n-1} + \alpha_{2n})\\
\times&(\mu\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{2n-2} + \alpha_{2n-1} + \lambda\alpha_{2n}).
\end{aligned}$$

When the multiplication has been performed we require the complete coefficient of

$$\alpha_1\alpha_2\alpha_3 \dots \alpha_{2n-2}\alpha_{2n-1}\alpha_{2n}.$$

Writing $\Sigma\alpha = h$, the tth factor of the product is

$$h + (\lambda - 1)\,\alpha_t + (\mu - 1)\,\alpha_{2n-t+1},$$

and since we only require the quantities α with unit exponents the product of the complementary factors

$$\begin{aligned}
&h + (\lambda - 1)\,\alpha_t + (\mu - 1)\,\alpha_{2n-t+1},\\
&h + (\lambda - 1)\,\alpha_{2n-t+1} + (\mu - 1)\,\alpha_t,
\end{aligned}$$

the second of these being the $(2n - t + 1)$th factor of the product, is effectively

$$h^2 + (\lambda + \mu - 2)\,(\alpha_t + \alpha_{2n-t+1})\,h + \{(\lambda - 1)^2 + (\mu - 1)^2\}\,\alpha_t\alpha_{2n-t+1},$$

and the complete product may now be written

$$\prod_{t=1}^{t=n}\,[h^2 + (\lambda + \mu - 2)\,(\alpha_t + \alpha_{2n-t+1})\,h + \{(\lambda - 1)^2 + (\mu - 1)^2\}\,\alpha_t\alpha_{2n-t+1}].$$

Consider these n factors of degree 2. Each one of them involves explicitly two of the quantities α, say α_t and α_{2n-t+1}, neither of which occurs explicitly in any of the other $n - 1$ factors. In the product of the n factors therefore each of the quantities α occurs explicitly to the power unity. Hence if we write the product in the form

$$h^{2n} + A_1h^{2n-1} + A_2h^{2n-2} + \dots + A_{2n},$$

A_m is a linear function of products of the quantities α, each of which contains m *different* quantities α, each with the exponent unity.

In particular the reader will readily observe that

$$\begin{aligned}
A_1 &= (\lambda + \mu - 2)\,\Sigma\alpha,\\
A_{2n} &= \{(\lambda - 1)^2 + (\mu - 1)^2\}^n\,\alpha_1\alpha_2 \dots \alpha_{2n-1}\alpha_{2n}.
\end{aligned}$$

Now $$h^m = \dots + m\,!\,\Sigma\alpha_1\alpha_2 \dots \alpha_{m-1}\alpha_m + \dots,$$

as is evident by operating with the differential operator D_1 of Section I m times successively.

This is the only portion of the development of h^m in which the exponents of the α's do not rise above unity. Hence for the present purpose

$$h^m = m\,!\,\Sigma\alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m \text{ effectively.}$$

The co-factor of h^m in the product is A_{2n-m} which is a sum of products of the quantities α, each of which contains $2n-m$ different quantities α (each with exponent unity) and has a certain function of λ and μ as a coefficient. Each of the α products which occurs in A_{2n-m} can be multiplied into one of the members of the sum

$$\Sigma\alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m$$

to form the product $\alpha_1\alpha_2\ldots\alpha_{2n-1}\alpha_{2n}$ because for this purpose we associate an α product occurring in A_{2n-m} with that member of $\Sigma\alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m$ which does not contain any α which occurs in the α product. That particular α product will not be multiplied into any other member of $\Sigma\alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m$ because that would produce an α with an exponent 2.

The result of the multiplication

$$A_{2n-m}h^m$$

is to produce as many effective terms as A_{2n-m} contains α products.

414. We may therefore, to obtain the desired coefficient, put in

$$A_{2n-m}h^m$$

each quantity α, that occurs explicitly in A_{2n-m}, equal to unity and h^m equal to $m\,!$. Hence therefore in

$$h^{2n} + A_1h^{2n-1} + A_2h^{2n-2} + \ldots + A_{2n},$$

we may put each α, occurring explicitly in the quantities A, equal to unity and $h^m = m\,!$ where m ranges from $2n$ to unity.

It is thus established that in the product

$$\prod_{t=1}^{t=n}\left[h^2 + (\lambda+\mu-2)(\alpha_t + \alpha_{2n-t+1})\,h + \{(\lambda-1)^2 + (\mu-1)^2\}\,\alpha_t\alpha_{2n-t+1}\right],$$

we may put each explicitly occurring α equal to unity and regard h as a symbol such that on completing the multiplication any power of h, say h^m, is to denote $m\,!$.

Thus the desired coefficient is expressed by

$$\{h^2 + 2(\lambda+\mu-2)\,h + (\lambda-1)^2 + (\mu-1)^2\}^n,$$

or by

$$\{h^2 - 4h + 2 + 2(\lambda+\mu)(h-1) + \lambda^2 + \mu^2\}^n,$$

where, after development, h^m is to be put equal to $m\,!$.

The coefficient is by this means represented symbolically as an nth power.

415. Putting $h^2 - 4h + 2 = \eta_2$, $h - 1 = \eta_1$ it is more conveniently written

$$\{\eta_2 + 2(\lambda + \mu)\eta_1 + \lambda^2 + \mu^2\}^n.$$

For an even order $2n$ the coefficient herein of

$$\lambda^l \mu^m$$

enumerates the squares which have one unit in each row and in each column, l units in the λ diagonal and m units in the μ diagonal.

For an uneven order $2n+1$, the $(n+1)$th factor corresponding to the $(n+1)$th row of the square is

$$h + (\lambda\mu - 1)\alpha_{n+1},$$

and thus the product of the $2n+1$ factors may be written

$$\prod_{t=1}^{t=n} [h^2 + (\lambda + \mu - 2)(\alpha_t + \alpha_{2n-t+2})h + \{(\lambda - 1)^2 + (\mu - 1)^2\}\alpha_t\alpha_{2n-t+2}]$$
$$\times \{h + (\lambda\mu - 1)\alpha_{n+1}\},$$

and now the coefficient of $\alpha_1\alpha_2 \ldots \alpha_{2n}\alpha_{2n+1}$ is, by the same argument, represented by the symbolic form

$$\{\eta_2 + 2(\lambda + \mu)\eta_1 + \lambda^2 + \mu^2\}^n (\eta_1 + \lambda\mu),$$

the complete solution in respect of the uneven order $2n + 1$.

416. We may test the truth of these results in two ways.

(i) If we put $\lambda = \mu = 1$ the coefficient is $(2n)!$ for even order and $(2n+1)!$ for uneven order. This is verified because our symbolic expressions become in this case

$$h^{2n} \text{ and } h^{2n+1} \text{ respectively.}$$

(ii) If we put $\lambda = 0$, $\mu = 1$, the coefficient is equal to the number of permutations of $\alpha_1\alpha_2 \ldots \alpha_{2n}$ or of $\alpha_1\alpha_2 \ldots \alpha_{2n+1}$ such that every quantity α_s is displaced. Our symbolic expressions become

$$(h-1)^{2n} \text{ and } (h-1)^{2n+1} \text{ respectively,}$$

equivalent to

$$(2n)! - \binom{2n}{1}(2n-1)! + \binom{2n}{2}(2n-2)! - \ldots,$$

$$(2n+1)! - \binom{2n+1}{1}(2n)! + \binom{2n+1}{2}(2n-1)! - \ldots,$$

which agree with the results of Section III, Chapter III, Art. 69.

417. To find the number of general magic squares corresponding to the sum unity we require the coefficient of $\lambda\mu$ and this is found to be

$$\text{for even order } 2n \qquad 8\binom{n}{2}\eta_2^{n-2}\eta_1^2,$$

$$\text{for uneven order } 2n+1 \quad 8\binom{n}{2}\eta_2^{n-2}\eta_1^3 + \eta_2^n.$$

Those squares which have only zeros in the diagonals are enumerated by η_2^n and $\eta_2^n\eta_1$ according as the order is even or uneven, while those which involve the maximum number of units in the diagonals depend merely upon the expressions

$$(\lambda^2+\mu^2)^n, \quad (\lambda^2+\mu^2)^n\lambda\mu.$$

418. The calculation of products of powers of η_2 and η_1 is simplified by the consideration that since

$$\eta_2 = h^2 - 4h + 2, \quad \eta_1 = h - 1,$$

we have $$\eta_2 = \eta_1^2 - 2\eta_1 - 1$$

and $$\eta_1^p\eta_2^q = \eta_1^{p+2}\eta_2^{q-1} - 2\eta_1^{p+1}\eta_2^{q-1} - \eta_1^p\eta_2^{q-1},$$

a convenient formula because the numerical values of powers of η_1 are known.

419. The following table gives the values of simple η products:

η_1					0				
η_1^2	η_2				1	0			
η_1^3	$\eta_1\eta_2$				2	0			
η_1^4	$\eta_1^2\eta_2$	η_2^2			9	4	4		
η_1^5	$\eta_1^3\eta_2$	$\eta_1\eta_2^2$			44	24	16		
η_1^6	$\eta_1^4\eta_2$	$\eta_1^2\eta_2^2$	η_2^3		265	168	116	80	
η_1^7	$\eta_1^5\eta_2$	$\eta_1^3\eta_2^2$	$\eta_1\eta_2^3$		1854	1280	920	672	
η_1^8	$\eta_1^6\eta_2$	$\eta_1^4\eta_2^2$	$\eta_1^2\eta_2^3$	η_2^4	14833	10860	8132	6176	4752

Each column is derived from the column to its left.

Thus by the formula

$$\eta_1^3\eta_2^2 = \eta_1^5\eta_2 - 2\eta_1^4\eta_2 - \eta_1^3\eta_2,$$
$$920 = 1280 - 2 \times 168 - 24.$$

SECTION IX

PARTITIONS IN TWO DIMENSIONS

CHAPTER I

INTRODUCTORY NOTIONS

420. In this Section we take up a particular generalization of the Theory of Partitions which arises directly from the improved definition of a partition of a unipartite number due to the importation of an assigned order, viz. the descending order, among the parts and the consequent basing of the theory upon that of linear Diophantine Inequalities.

The Definition regards the parts of the partition as being placed at points of a line

$$\bullet \geqslant \bullet \geqslant \bullet \geqslant \bullet \geqslant \bullet \geqslant \bullet$$

and the symbol $\geqslant$ as regulating the magnitudes of the parts at any two adjacent points. It is important to realise that the partitions may be regarded as *partitions on a line* or in *one dimension* of space. Moreover the enumerating generating function

$$\frac{(i+1)(i+2)\dots(i+j)}{(1)(2)\dots(j)},$$

when the number of parts is limited by the number $\overset{i}{\underset{j}{.}}$ and the part magnitude by the number $\overset{j}{\underset{i}{.}}$, can also be associated with either i or j points upon the line. For example we take i points in a line

$$\frac{(j+1)}{(1)}\;\frac{(j+2)}{(2)}\;\frac{(j+3)}{(3)}\cdots\frac{(j+i)}{(i)},$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

and place the algebraic fractions

$$\frac{(j+1)}{(1)}, \frac{(j+2)}{(2)}, \frac{(j+3)}{(3)}, \ldots \frac{(j+i)}{(i)}$$

at the successive points from left to right, and we observe that the continued product of such fractions is in fact the generating function. Instead of so proceeding we may take j points in a line and again we find the generating function as the product of j algebraic fractions each of which is placed at one of the j points, viz.

$$\frac{(i+1)}{(1)} \frac{(i+2)}{(2)} \frac{(i+3)}{(3)} \cdots \frac{(i+j)}{(j)}.$$

....

421. We are about to see that we may have partitions in two dimensions of space and it is interesting to observe that the Ferrers-Sylvester graph of a partition of a unipartite number is in reality a partition in two dimensions. Such a graph of the partition 4311 is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \\ \bullet & & & \end{array}$$

and we have already seen that for many purposes it is advantageous to replace the nodes by units, leading to the unit graph

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & & & \\ 1 & & & \end{array}$$

which as we shall see is a two-dimensional partition of the number 9 in agreement with the definition which is now given.

Consider the points of a two-dimensional lattice

$$\begin{array}{ccccccccccc} O & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & x \\ & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & \\ & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \\ & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & \\ & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \\ & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & & \vee\!\!\!\!\!\!\!\;\backslash & \\ & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \geqslant & \bullet & \\ & y & & & & & & & & & \end{array}$$

and let the first row and first column be axes of x and y respectively.

Suppose the parts of the partition to be placed at these points so that a descending order of magnitude is in evidence in each row in direction of the x-axis and also in each column in direction of the y-axis.

The arrangement of numbers thus reached is defined to be a two-dimensional partition of the number partitioned.

Clearly the unit graph of a partition is a two-dimensional partition in which the part magnitude is limited not to exceed unity.

422. In the case of a partition in two dimensions we are concerned with three limiting numbers, for we may limit

(i) the number of rows,

(ii) the number of columns,

(iii) the part magnitude.

Suppose for a moment that there is no restriction upon the rows and columns but that the part magnitude is limited by the number l.

Guided by the results reached in the case of partitions on a line we may conjecture that the *form* of the enumerating generating function of the partitions will be

$$\frac{(l+1)^{s_1}}{(1)^{s_1}} \cdot \frac{(l+2)^{s_2}}{(2)^{s_2}} \cdot \frac{(l+3)^{s_3}}{(3)^{s_3}} \cdots.$$

If this be a right conjecture the function, on putting $l = 1$, is

$$\frac{1}{(1)^{s_1}\,(2)^{s_2-s_1}\,(3)^{s_3-s_2}\,(4)^{s_4-s_3}\ldots \text{ ad inf.}}.$$

But if $l = 1$, the two-dimensional partitions enumerated are those of the nature

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & & & \\ 1 & & & \end{array}$$

where there is no restriction upon the rows and columns. These are the unit graphs of the one-dimensional partitions of all numbers and therefore are enumerated by

$$\frac{1}{(1)\,(2)\,(3)\,(4)\ldots \text{ ad inf.}}.$$

Comparison now gives

$$s_1 = 1, \quad s_2 = 2, \quad s_3 = 3, \ \ldots\ s_p = p, \ \ldots,$$

establishing that, on the assumption as to form, the generating function which enumerates two-dimensional partitions, with the single restriction that the part magnitude is not to exceed l, is

$$\frac{(l+1)}{(1)} \cdot \frac{(l+2)^2}{(2)^2} \cdot \frac{(l+3)^3}{(3)^3} \cdot \frac{(l+4)^4}{(4)^4} \ldots \text{ ad inf.}$$

This is, in fact, the result which will be rigidly established later.

423. It is to be particularly noted that this result also can be depicted upon the lattice in the manner following:

$$\begin{array}{cccccc}
O & \frac{(l+1)}{(1)} & \frac{(l+2)}{(2)} & \frac{(l+3)}{(3)} & \frac{(l+4)}{(4)} & \cdot\;\cdot \quad x \\[1ex]
& \frac{(l+2)}{(2)} & \frac{(l+3)}{(3)} & \frac{(l+4)}{(4)} & \frac{(l+5)}{(5)} & \cdot\;\cdot \\[1ex]
& \frac{(l+3)}{(3)} & \frac{(l+4)}{(4)} & \frac{(l+5)}{(5)} & \frac{(l+6)}{(6)} & \cdot\;\cdot \\[1ex]
& \cdot & \cdot & \cdot & \cdot & \\
& y & & & &
\end{array}$$

We place $\frac{(l+1)}{(1)}$ at the origin and give a unit increase whenever we advance a segment in the direction of either axis.

In future we shall speak of a plane partition instead of using the combined word two-dimensional.

As an example of a plane partition take

$$\begin{array}{lll}
3 & 3 & 1 - x \\
2 & 2 & \\
1 & & \\
| & & \\
y & &
\end{array}$$

Note that we may replace each part by a pile of nodes in the direction of the z-axis and that we thus arrive at a solid graph analogous to the Ferrers-Sylvester plane graph.

The enumeration of plane partitions is the same as the enumeration of the solid graphs. In the graph the three restricting numbers limit the numbers of nodes that may appear along the three axes of the graph and we have a more direct symmetry, which is at once apparent in plane partitions. If l, m, n be the three restricting numbers we must expect the general generating function to remain unaltered when l, m, n are subjected to any permutation; just as in line partitions the numbers limiting the number of parts and the part magnitude are interchangeable.

In the solid graph we may substitute units for nodes and thus realise a *solid partition* in which the part magnitude is limited by unity. We may say that there is a one-to-one correspondence between plane partitions and solid partitions in which the part magnitude is limited by unity. The plane partition arises by projection of the solid partition upon one of the coordinate planes, just as the line partition arises by projection of the plane partition (with which it is in correspondence) upon one of the coordinate axes.

424. The reader will find it interesting at this point to make the following observation. Consider for a moment solid partitions unrestricted in regard to the three axes but such that the magnitude of the part at the origin is limited not to exceed k. The descending order of magnitude is to be in evidence in the direction of each axis, so that it is only necessary to restrict the magnitude of the part at the origin.

Assume the *form* of the enumerating generating function to be

$$\frac{(\mathbf{k+1})^{r_1}}{(\mathbf{1})^{r_1}} \cdot \frac{(\mathbf{k+2})^{r_2}}{(\mathbf{2})^{r_2}} \cdot \frac{(\mathbf{k+3})^{r_3}}{(\mathbf{3})^{r_3}} \cdots.$$

This for $k=1$ becomes

$$\frac{1}{(\mathbf{1})^{r_1}(\mathbf{2})^{r_2-r_1}(\mathbf{3})^{r_3-r_2}\ldots \text{ ad inf.}},$$

and enumerates the solid graphs in which the part magnitude does not exceed unity; and these, being equi-numerous with plane partitions which are absolutely unrestricted, are enumerated by

$$\frac{1}{(\mathbf{1})(\mathbf{2})^2(\mathbf{3})^3(\mathbf{4})^4\ldots \text{ ad inf.}}.$$

Hence by comparison

$$r_1=1,\ r_2=3,\ r_3=6,\ \ldots r_p=\binom{p+1}{2},$$

and on the assumption as to form the generating function of the solid partitions we are considering must be

$$\frac{(\mathbf{k+1})}{(\mathbf{1})} \cdot \frac{(\mathbf{k+2})^3}{(\mathbf{2})^3} \cdot \frac{(\mathbf{k+3})^6}{(\mathbf{3})^6} \cdots \frac{(\mathbf{k+p})^{\binom{p+1}{2}}}{(\mathbf{p})^{\binom{p+1}{2}}} \ldots \text{ ad inf.*}$$

Moreover, we notice again that this expression can be depicted on the points of the solid graph just as in the plane case.

In the cases of line, plane and solid partitions the enumerating functions appear to be associated with the series of numbers

line	1,	1,	1,	1, ...
plane	1,	2,	3,	4, ...
solid	1,	3,	6,	10, ...

viz. with the successive orders of figurate numbers.

If we go *back* a step to point partitions it is clear that the function is

$$\frac{(\mathbf{l+1})}{(\mathbf{1})},$$

yielding the series of numbers

$$1,\ 0,\ 0,\ 0,\ \ldots,$$

* This assumption as to form is shewn later not to be justified.

which is the series of figurate numbers prior to

$$1, \quad 1, \quad 1, \quad 1, \ldots .$$

It may be remarked that the function

$$\frac{(k+1)}{(1)} \cdot \frac{(k+2)^3}{(2)^3} \cdot \frac{(k+3)^6}{(3)^6} \cdots \frac{(k+p)^{\binom{p+1}{2}}}{(p)^{\binom{p+1}{2}}} \ldots \text{ ad inf.}$$

may be also written as the product of

$$\frac{1}{(1)\,(2)^3\,(3)^6 \ldots (k)^{\binom{k+1}{2}}},$$

and

$$\frac{1}{(k+1)^{\frac{1}{2}k(k+3)}\,(k+2)^{\frac{1}{2}k(k+5)}\,(k+3)^{\frac{1}{2}k(k+7)} \ldots \text{ ad inf.}}$$

425. Before broaching the formal study of the enumeration of plane partitions we must look into the properties of the solid graphs to which they give rise and enquire into their interpretation.

The graph of a partition of a unipartite number, say

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & & \end{array}$$

viz. the partition 642 of the number 12, may be also regarded as the graph of the multipartite number $\overline{642}$.

For present purposes we will regard it as being such.

If we take the successive layers, in the plane xy, of a solid graph we obtain a succession of multipartite numbers in correspondence. This succession may be regarded as constituting the successive parts of the partition of a multipartite number. Reading the graph by successive layers in either of the three coordinate planes we thus obtain a partition of a multipartite number.

Moreover in each of the three coordinate planes the layers may be read in the direction of either of two axes, so that there are altogether six (3 !) ways in which the graph may be read and in each case we arrive at a partition of a multipartite number.

But conversely it is not true that every partition of a multipartite number may be represented by a *regular* solid graph.

For suppose

$$(\overline{a_1 a_2 a_3 \ldots}, \quad \overline{b_1 b_2 b_3 \ldots}, \quad \overline{c_1 c_2 c_3 \ldots}, \ldots)$$

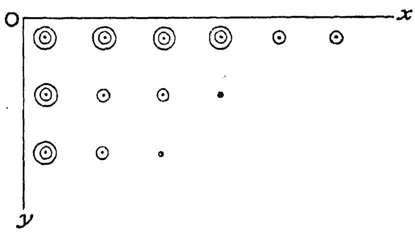
to be a multipartite partition. In order to arrive at a *regular* solid graph the series of numbers

$$\begin{array}{ll} a_1,\ a_2,\ a_3,\ \ldots & a_1,\ b_1,\ c_1,\ \ldots \\ b_1,\ b_2,\ b_3,\ \ldots & a_2,\ b_2,\ c_2,\ \ldots \\ c_1,\ c_2,\ c_3,\ \ldots & a_3,\ b_3,\ c_3,\ \ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array}$$

must all be in descending order of magnitude.

It follows that the multipartite partitions that are in evidence in the greater part of the present investigation are those which are representable by *regular* solid graphs.

Consider now the solid graph



$$\equiv \begin{array}{l} 333322 \\ 3221 \\ 321 \end{array}$$

the axis of z being perpendicular to the plane of the paper.

We read as follows:

Planes parallel to the plane xy and in direction Ox

$$(\overline{643},\ \overline{632},\ \overline{411}) \text{ a partition of } (\overline{16, 8, 6}).$$

Planes parallel to the plane xy and in direction Oy

$$(\overline{333211},\ \overline{332111},\ \overline{311100}) \text{ a partition of } (\overline{976422}).$$

Planes parallel to the plane yz and in direction Oy

$$(\overline{333},\ \overline{331},\ \overline{321},\ \overline{211},\ \overline{110},\ \overline{110}) \text{ a partition of } (\overline{13, 11, 6}).$$

Planes parallel to the plane yz and in direction Oz

$$(\overline{333},\ \overline{322},\ \overline{321},\ \overline{310},\ \overline{200},\ \overline{200}) \text{ a partition of } (\overline{16, 8, 6}).$$

Planes parallel to the plane xz and in direction Oz

$$(\overline{333322},\ \overline{322100},\ \overline{321000}) \text{ a partition of } (\overline{976422}).$$

Planes parallel to the plane xz and in direction Ox

$$(\overline{664},\ \overline{431},\ \overline{321}) \text{ a partition of } (\overline{13, 11, 6}).$$

The graph is representative of three multipartite numbers and of two partitions of each. The readings that give partitions of the multipartite $\overline{16, 8, 6}$ are those (i) in plane xy and in direction Ox, (ii) in plane yz and in direction Oz. It will be noted that in both instances the specification of the plane and the direction in the plane involve the letter y once only. It is for

this reason that the two readings are associated with the same multipartite number. So also the readings (i) in the plane xy and in direction Oy, (ii) in the plane xz and in direction Oz, involving in their specifications the letter x once only, refer to the same multipartite number $\overline{976422}$. The remaining two readings involve the letter z once only and refer to $\overline{13, 11, 6}$.

In regard to the six partitions, if one of them contains a highest figure equal to p, is r-partite and has q parts and we say that r, p, q are its characteristics, then the other five partitions will have the characteristics

$$\begin{array}{ccc} p & r & q \\ q & r & p \\ r & q & p \\ p & q & r \\ q & p & r \end{array}$$

derived from the permutations of p, q, r.

The two partitions which are r-partite appertain to the same multipartite number; similarly for the pairs which are p-partite and q-partite respectively. Hence the three multipartite numbers involved correspond to the three pairs of permutations so formed that in any pair the commencing symbol of each permutation is the same.

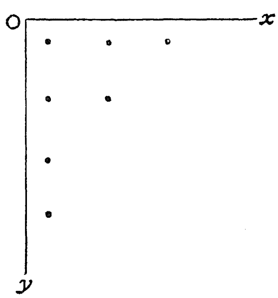
426. The consideration of graphs formed with a given number of nodes now leads to the theorem:

"The enumeration of the graphically regularized r-partite partitions into q parts and having p for the highest figure gives the same number for each of the six ways in which the numbers p, q, r may be permuted."

The six-fold conjugation of the characteristics may obtain even though there may be equalities between the numbers p, q, r.

It is interesting to view the plane graphs of unipartite numbers from the point of view of the solid graph.

Take the graph



which in the unipartite theory denotes the partition 3211 when read in direction Ox and 421 when read in direction Oy.

In the present theory there are six readings:

Plane xy	Direction Ox	$(\overline{3211})$	$p,\ q,\ r = 3,\ 1,\ 4$
,, xy	,, Oy	$(\overline{421})$	,, $= 4,\ 1,\ 3$
,, yz	,, Oy	(421)	,, $= 4,\ 3,\ 1$
,, yz	,, Oz	$(\overline{1111}\ \overline{1100}\ \overline{1000})$	,, $= 1,\ 3,\ 4$
,, zx	,, Oz	(3211)	,, $= 3,\ 4,\ 1$
,, zx	,, Ox	$(\overline{111}\ \overline{110}\ \overline{100}\ \overline{100})$	,, $= 1,\ 4,\ 3$

The three multipartite numbers (7), $(\overline{421})$, $(\overline{3211})$ appear each in two partitions.

In general if the plane graph is of a partition involving a highest part p and q parts, we establish a 6-fold correspondence between

(i)	a unipartite partition	having a highest part	p and	q parts.
(ii)	,, ,,	,, ,,	q ,,	p ,,
(iii)	a p-partite partition	,, ,,	1 ,,	q ,,
(iv)	,, ,,	,, ,,	q ,,	1 part.
(v)	a q-partite partition	,, ,,	p ,,	1 ,,
(vi)	,, ,,	,, ,,	1 ,,	p parts.

427. It has been shewn that a graph admits in general of six readings. In other words we may say that a graph has six aspects depending upon the position from which it is viewed. Thus, denoting by the integer m a pile of nodes (or units) m nodes high, the graph examined in Art. 425 is denoted by

333322
3221
321

and the other five aspects it can assume are

333211	333	333	664	643
332111	331	322	431	632
3111	321	321	321	411
	211	31		
	11	2		
	11	2		

A graph however does not always admit of six different readings or assume six different aspects.

Thus the graph

⊙ • or 21
• 1

admits of only one distinct reading, viz.

$$(\overline{21}\ \overline{10}),$$

and has no other aspect. It is in fact completely symmetrical.

This is also obviously the property of the graph which consists of a single node, and these two are the earliest of an infinite series of completely symmetrical graphs.

Again, the graph

$$\bullet\ \ \bullet \quad \text{or} \quad 11$$

admits of three distinct readings, viz.

$$(\overline{11}),\quad (2),\quad (11),$$

and of three aspects. This is the first of an infinite series of graphs which have three different readings and three aspects.

428. There is also an infinite series of graphs which have two distinct readings and two aspects. The simplest of these involves thirteen nodes and is

$$\begin{matrix} 331 \\ 211 \\ 2 \end{matrix}$$

the other aspect is

$$\begin{matrix} 322 \\ 31 \\ 11 \end{matrix}$$

and the two readings are

$$(\overline{331}\ \overline{211}\ \overline{110}),\quad (\overline{322}\ \overline{310}\ \overline{110}).$$

Each of the solid graphs therefore belongs to one of four classes according as it has 1, 2, 3 or 6 distinct readings and can assume 1, 2, 3 or 6 aspects.

Graphs which are different aspects of the same graph are not essentially different. Ex. gr. of two nodes there are three graphs

$$11,\quad \begin{matrix} 1, \\ 1 \end{matrix}\quad 2$$

but they are all aspects of any one of them, so that of two nodes there is only one essentially distinct graph.

In examining or tabulating the graphs appertaining to a given number of nodes we may or may not take account of the essentially distinct graphs.

Thus the thirteen different graphs that can be formed of four nodes may be written down by noting that the really distinct graphs are

$$1111,\quad \begin{matrix} 111, \\ 1 \end{matrix}\quad \begin{matrix} 21, \\ 1 \end{matrix}\quad \begin{matrix} 11 \\ 11 \end{matrix}$$

and we find

Graphs	1111	111 1	21 1	11 11
Readings	(4) $(\overline{1111})$ (1111)	$(\overline{31})$ (31) (211) (211) $(\overline{111}\ \overline{100})$ $(\overline{11}\ \overline{10}\ \overline{10})$	$(\overline{21}\ \overline{10})$	$(\overline{22})$ (22) $(\overline{11}\ \overline{11})$

or we may write down the whole of the graphs and interpret each by reading in the plane xy and in the direction Ox. Thus

Graphs	1111	111 1	211	11 1 1	21 1	11 11	31
Readings	(4)	$(\overline{31})$	(31)	$(\overline{211})$	$(\overline{21}\ \overline{10})$	(22)	(211)

Graphs	22	1 1 1 1	2 1 1	3 1	2 2	4
Readings	(22)	$(\overline{1111})$	$(\overline{111}\ \overline{100})$	$(\overline{11}\ \overline{10}\ \overline{10})$	$(\overline{11}\ \overline{11})$	(1111)

Some remarks will be made later as to the nature of the symmetry which results in a graph possessing one, two or three aspects.

No one, so far, has succeeded in enumerating the number of really distinct graphs that appertain to a given number of nodes.

Omitting the trivial graph of unity every plane graph can be read in three or six ways. Those which have three readings may be

(i) those which have a single line *or* a single column of nodes,

(ii) those which are self-conjugate.

The remainder have six readings.

CHAPTER II

THE METHOD OF DIOPHANTINE INEQUALITIES

429. The problem of the enumeration of solid graphs may be regarded as a question of unipartite partition in two dimensions. It may be given the following statement.

Integers limited in magnitude to n, zero being included, are arranged in a rectangle of m rows and l columns in such wise that in each row and in each column there is a descending order of magnitude. It is required to find the number of ways in which a given number w can be partitioned into integers which satisfy these conditions.

As soon as we import the idea of the descending order of magnitude into ordinary or line partitions this generalization to plane partitions at once comes into view.

We denote by $$GF(l\,;m\,;n)$$
a function of x such that the coefficient of x^w gives the number of arrangements.

It will be convenient to give at once the result that will be established later:

$$\begin{aligned} &GF(l\,;m\,;n)\\ &=\frac{(1-x^{n+1})}{(1-x)}\cdot\frac{(1-x^{n+2})^2}{(1-x^2)^2}\cdots\frac{(1-x^{n+l-1})^{l-1}}{(1-x^{l-1})^{l-1}}\\ &\times\left\{\frac{(1-x^{n+l})}{(1-x^l)}\cdot\frac{(1-x^{n+l+1})}{(1-x^{l+1})}\cdots\frac{(1-x^{n+m})}{(1-x^m)}\right\}^l\\ &\times\frac{(1-x^{n+m+1})^{l-1}}{(1-x^{m+1})^{l-1}}\cdot\frac{(1-x^{n+m+2})^{l-2}}{(1-x^{m+2})^{l-2}}\cdots\frac{(1-x^{n+m+l-1})}{(1-x^{m+l-1})}, \end{aligned}$$

wherein m is supposed to be $\geqslant l$.

This supposition does not affect the generality of the formula because it is symmetrical in the three magnitudes l, m, n. These therefore can be arranged in any desired order. The reader will have no difficulty in establishing the symmetry referred to.

The expression above written involves lm factors in numerator and in denominator. It may also be written as a product of mn or of nl factors, by permuting the numbers l, m, n.

In the usual notation $1 - x^s = (s)$ it may also be written

$$\left\{\frac{(n+1)}{(1)} \cdot \frac{(n+2)}{(2)} \cdots \frac{(n+m+l-1)}{(m+l-1)}\right\}$$
$$\times \left\{\frac{(n+2)}{(2)} \cdot \frac{(n+3)}{(3)} \cdots \frac{(n+m+l-2)}{(m+l-2)}\right\}$$
$$\times \left\{\frac{(n+3)}{(3)} \cdot \frac{(n+4)}{(4)} \cdots \frac{(n+m+l-3)}{(m+l-3)}\right\}$$

$\times \ldots$ to l factors in brackets $\{\ \}$.

430. To begin with the most simple case. The numbers are placed at the corners of a square

$$\begin{array}{ccc} \alpha_{11}\bullet & \geqslant & \bullet\alpha_{12} \\ \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} & & \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} \\ \alpha_{21}\bullet & \geqslant & \bullet\alpha_{22} \end{array}$$

subject to the descending orders as shewn. If there is no limit to the magnitude of the numbers this is the case $(l;m;n) = (2;2;\infty)$ and we find, employing the auxiliaries $\lambda_1, \lambda_2, \lambda_3, \lambda_4$,

$$GF(\infty;2;2) = \underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 x_1)\left(1-\dfrac{\lambda_3}{\lambda_2}x_2\right) \times \left(1-\dfrac{\lambda_4}{\lambda_1}x_3\right)\left(1-\dfrac{1}{\lambda_3\lambda_4}x_4\right)},$$

wherein subsequently we are to put $x_1 = x_2 = x_3 = x_4 = x$.

Eliminating λ_1 and λ_3 this is

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_2 x_1)\left(1-\dfrac{1}{\lambda_2}x_2\right) \times (1-\lambda_2\lambda_4 x_1 x_3)\left(1-\dfrac{1}{\lambda_2\lambda_4}x_2x_4\right)},$$

and eliminating λ_4 and λ_2 in succession we find

$$\frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},$$

a real generating function shewing the fundamental partitions

$$\begin{matrix}1&0\\0&0\end{matrix}\qquad\begin{matrix}1&1\\0&0\end{matrix}\qquad\begin{matrix}1&0\\1&0\end{matrix}\qquad\begin{matrix}1&1\\1&0\end{matrix}\qquad\begin{matrix}1&1\\1&1\end{matrix}$$

from which all partitions can be obtained by addition, and the syzygy

$$\begin{matrix}1&0\\0&0\end{matrix} + \begin{matrix}1&1\\1&0\end{matrix} = \begin{matrix}1&1\\0&0\end{matrix} + \begin{matrix}1&0\\1&0\end{matrix}$$

Putting $x_1 = x_2 = x_3 = x_4 = x$ we obtain the enumerating function

$$GF(2;2;\infty) = \frac{1}{(1)(2)^2(3)}.$$

431. Also

$$GF(2;2;n)=\underset{\geqslant}{\Omega}\frac{1-(\lambda_1\lambda_2 x)^{n+1}}{(1-\lambda_1\lambda_2 x).\left(1-\frac{\lambda_3}{\lambda_2}x\right)\times\left(1-\frac{\lambda_4}{\lambda_1}x\right).\left(1-\frac{1}{\lambda_3\lambda_4}x\right)},$$

and since

$$\frac{1}{(1-g)(1-g\lambda_1\lambda_2 x)}=\sum_0^\infty g^n\frac{1-(\lambda_1\lambda_2 x)^{n+1}}{1-\lambda_1\lambda_2 x},$$

we require the coefficient of g^n in

$$\underset{\geqslant}{\Omega}\frac{1}{(1-g)(1-g\lambda_1\lambda_2 x)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\times\left(1-\frac{\lambda_4}{\lambda_1}x\right)\left(1-\frac{1}{\lambda_3\lambda_4}x\right)},$$

which is readily seen to reduce to

$$\frac{1+gx^2}{(1-g)(1-gx)(1-gx^2)(1-gx^3)(1-gx^4)}.$$

Now we know from the Section on line partitions that

$$\frac{1}{(1-g)(1-gx)(1-gx^2)(1-gx^3)(1-gx^4)}$$
$$=\sum_0^\infty g^n\frac{(\mathrm{n}+1)(\mathrm{n}+2)(\mathrm{n}+3)(\mathrm{n}+4)}{(1)(2)(3)(4)}.$$

Hence the expression becomes

$$\sum_0^\infty g^n\left\{\frac{(\mathrm{n}+1)(\mathrm{n}+2)(\mathrm{n}+3)(\mathrm{n}+4)}{(1)(2)(3)(4)}+x^2\frac{(\mathrm{n})(\mathrm{n}+1)(\mathrm{n}+2)(\mathrm{n}+3)}{(1)(2)(3)(4)}\right\}$$
$$=\sum_0^\infty g^n\frac{(\mathrm{n}+1)(\mathrm{n}+2)^2(\mathrm{n}+3)}{(1)(2)^2(3)},$$

and we have the result

$$GF(2;2;n)=\frac{(\mathrm{n}+1)(\mathrm{n}+2)^2(\mathrm{n}+3)}{(1)(2)^2(3)}.$$

432. We shall now obtain this result in another manner by considering the graphs $(\infty;2;2)$ and $(n;2;2)$. In the case of the former we have two rows of infinite length and a part magnitude not exceeding 2. The partitions are such as

2222111...

22111...

which have the property that in the first row the number of twos is equal to or greater than the number of twos in the second row, and also that the number of twos added to the number of ones in the first row is equal to or greater than the similar sum in the second row. If λ_4 and λ_2 be the auxiliaries appertaining to these inequalities respectively we are led to the crude enumerating function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_2 x)\left(1-\frac{1}{\lambda_2}x\right) \times (1-\lambda_2\lambda_4 x^2)\left(1-\frac{1}{\lambda_2\lambda_4}x^2\right)},$$

and this leads by the foregoing article to the same result as before.

If the rows are not to contain more than n numbers the crude function on expansion must not involve λ_2 to a higher power than n in the portion

$$\frac{1}{(1-\lambda_2 x)(1-\lambda_2\lambda_4 x^2)}$$

of the function. We are therefore led to the function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-g)(1-g\lambda_2 x)(1-g\lambda_2\lambda_4 x^2)\left(1-\frac{x}{\lambda_2}\right)\left(1-\frac{1}{\lambda_2\lambda_4}x^2\right)},$$

in which we take the coefficient of g^n.

This leads to the result already obtained.

433. Instead of dealing with the numbers 2 and 1 in the two rows we may more generally deal with the numbers p and q, and if there be no restriction on the length of the rows this leads to the function

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_2 x^q)\left(1-\frac{1}{\lambda_2}x^q\right) \times (1-\lambda_2\lambda_4 x^p)\left(1-\frac{1}{\lambda_2\lambda_4}x^p\right)},$$

which is readily expressed in the form

$$\frac{1-x^{p+2q}}{(1-x^p)(1-x^q)(1-x^{2p})(1-x^{p+q})(1-x^{2q})}.$$

In particular if $p=q=1$, this becomes

$$\frac{(3)}{(1)^2(2)^3},$$

which on development $= 1 + 2x + 6x^2 + 9x^3 + 18x^4 + \ldots$.

The partitions enumerated involve two rows of units of two different kinds.

Let u and v denote these units, then of the content three the nine partitions given by the formula are

$$\begin{array}{ccccccccc} uuu & uu & uuv & uu & uv & uv & uvv & vvv & vv \\ & u & & v & u & v & & & v \end{array}$$

the u unit, when it is present, always precedes the v unit in row and in column.

433[bis]. Now consider the system of partitions $(3\,;\,2\,;\,n)$.

An investigation carried out in the same manner as that above results in the generating function

$$\frac{(\mathrm{n}+1)\,(\mathrm{n}+2)^2\,(\mathrm{n}+3)^2\,(\mathrm{n}+4)}{(1)\,(2)^2\,(3)^2\,(4)},$$

and also in the formula

$$\Sigma g^n\, GF(3\,;\,2\,;\,n) = \Sigma g^n\,\frac{(\mathrm{n}+1)\,(\mathrm{n}+2)^2\,(\mathrm{n}+3)^2\,(\mathrm{n}+4)}{(1)\,(2)^2\,(3)^2\,(4)}$$

$$= \frac{1+g\,(x^2+x^3+x^4)+g^2x^6}{(1-g)\,(1-gx)\,(1-gx^2)\,(1-gx^3)\,(1-gx^4)\,(1-gx^5)\,(1-gx^6)},$$

a result which is particularly recommended to the reader's attention for the reason that it is fundamentally connected with much that is given later in the work and is fully generalizable.

434. There is no difficulty in forming a crude generating function for the general case $(l\,;\,m\,;\,n)$. The denominator is a rectangle of either lm, mn or nl factors at pleasure. The most convenient course is to choose the rectangle corresponding to the smallest of these three numbers. We choose to take, let us say, lm factors and form the rectangle of l columns and m rows. There will then be $(l-1)\,m$ auxiliaries in the crude function if we form it according to the second method employed above. Observe that using the first method there would be $(l-1)\,m+(m-1)\,l$ auxiliaries of which $(m-1)\,l$ could be eliminated at sight. The crude function is

$$\underset{\geqslant}{\Omega}\ \frac{1}{\begin{array}{c}(1-\lambda_1 x)\left(1-\frac{\lambda_2}{\lambda_1}x\right)\left(1-\frac{\lambda_3}{\lambda_2}x\right)\cdots\left(1-\frac{\lambda_{l-1}}{\lambda_{l-2}}x\right)\left(1-\frac{1}{\lambda_{l-1}}x\right)\\ (1-\lambda_1\mu_1x^2)\left(1-\frac{\lambda_2\mu_2}{\lambda_1\mu_1}x^2\right)\left(1-\frac{\lambda_3\mu_3}{\lambda_2\mu_2}x^2\right)\cdots\left(1-\frac{\lambda_{l-1}\mu_{l-1}}{\lambda_{l-2}\mu_{l-2}}x^2\right)\left(1-\frac{1}{\lambda_{l-1}\mu_{l-1}}x^2\right)\\ (1-\lambda_1\mu_1\nu_1x^3)\left(1-\frac{\lambda_2\mu_2\nu_2}{\lambda_1\mu_1\nu_1}x^3\right)\left(1-\frac{\lambda_3\mu_3\nu_3}{\lambda_2\mu_2\nu_2}x^3\right)\cdots\left(1-\frac{\lambda_{l-1}\mu_{l-1}\nu_{l-1}}{\lambda_{l-2}\mu_{l-2}\nu_{l-2}}x^3\right)\left(1-\frac{1}{\lambda_{l-1}\mu_{l-1}\nu_{l-1}}x^3\right)\\ \text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \cdots \qquad \text{,,} \qquad \text{,,}\\ \text{,,} \qquad \text{,,} \qquad \text{,,} \qquad \cdots \qquad \text{,,} \qquad \text{,,}\\ (1-\lambda_1\mu_1\ldots x^m)\left(1-\frac{\lambda_2\mu_2\ldots}{\lambda_1\mu_1\ldots}x^m\right)\left(1-\frac{\lambda_3\mu_3\ldots}{\lambda_2\mu_2\ldots}x^m\right)\cdots\left(1-\frac{\lambda_{l-1}\mu_{l-1}\ldots}{\lambda_{l-2}\mu_{l-2}\ldots}x^m\right)\left(1-\frac{1}{\lambda_{l-1}\mu_{l-1}\ldots}x^m\right)\end{array}}$$

It enumerates the partitions $(l\,;\,m\,;\,\infty)$.

To obtain the enumeration of the partitions $(l\,;\, m\,;\, n)$ it is merely necessary to write instead of the first column of factors, the column

$$\begin{array}{l}(1-g\lambda_1 x)\\(1-g\lambda_1\mu_1 x^2)\\(1-g\lambda_1\mu_1\nu_1 x^3)\\\qquad\vdots\\(1-g\lambda_1\mu_1\ldots x^m),\end{array}$$

and also to take the additional factor $1-g$. The generating function is the coefficient of g^n in the crude function.

Our knowledge of the $\underset{\geqslant}{\Omega}$ operation is not sufficient to enable us to establish the final form of result. This will be accomplished by the aid of new ideas which will be brought forward in the following chapters.

435. In the meantime it is necessary to impress upon the mind the nature of the result by shewing how it is directly connected with complete lattices. It is formed as a product of factors placed at the points of a lattice l nodes in breadth and m nodes in depth in the following manner:

$$\begin{array}{cccccc}
\frac{(n+1)}{(1)} & \frac{(n+2)}{(2)} & \frac{(n+3)}{(3)} & \cdots\cdots & \frac{(n+l-1)}{(l-1)} & \frac{(n+l)}{(l)}\\[2ex]
\frac{(n+2)}{(2)} & \frac{(n+3)}{(3)} & \frac{(n+4)}{(4)} & \cdots\cdots & \frac{(n+l)}{(l)} & \frac{(n+l+1)}{(l+1)}\\[2ex]
\frac{(n+3)}{(3)} & \frac{(n+4)}{(4)} & \frac{(n+5)}{(5)} & \cdots\cdots & \frac{(n+l+1)}{(l+1)} & \frac{(n+l+2)}{(l+2)}\\[2ex]
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\[2ex]
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\[2ex]
\frac{(n+m-1)}{(m-1)} & \frac{(n+m)}{(m)} & \frac{(n+m+1)}{(m+1)} & \cdots\cdots & \frac{(n+m+l-3)}{(m+l-3)} & \frac{(n+m+l-2)}{(m+l-2)}\\[2ex]
\frac{(n+m)}{(m)} & \frac{(n+m+1)}{(m+1)} & \frac{(n+m+2)}{(m+2)} & \cdots\cdots & \frac{(n+m+l-2)}{(m+l-2)} & \frac{(n+m+l-1)}{(m+l-1)}
\end{array}$$

In this tableau of factors the magnitudes l, m, n may be permuted in any manner so that there are six representations at the points of a complete lattice, two for each of the dimensions $l\times m$, $m\times n$, $n\times l$. Of these six, only three are essentially different, the remaining three being mere rotations of them. Whenever one of the numbers l, m, n is unity the partitions can be denoted by plane partitions composed entirely of units as well as by two line

partitions. Thus the system (3; 2; 1) has a generating function representable in the three ways:

$$\text{(i)} \quad \frac{\dfrac{(2)}{(1)} \cdot \dfrac{(3)}{(2)} \cdot \dfrac{(4)}{(3)}}{\dfrac{(3)}{(2)} \cdot \dfrac{(4)}{(3)} \cdot \dfrac{(5)}{(4)}}; \ l=3,\ m=2,\ n=1,$$

$$\text{(ii)} \quad \frac{(3)}{(1)} \cdot \frac{(4)}{(2)} \cdot \frac{(5)}{(3)}; \ l=3,\ m=1,\ n=2,$$

$$\text{(iii)} \quad \frac{(4)}{(1)} \cdot \frac{(5)}{(2)}; \ l=2,\ m=1,\ n=3,$$

the resulting product being $\dfrac{(4)\,(5)}{(1)\,(2)}$ in each case.

CHAPTER III

THE METHOD OF LATTICE FUNCTIONS

436. For the further discussion of the subject of the last chapter we introduce ideas derived from Vol. I, Sect. III, Chs. V and VI.

Therein we defined Lattice Permutations and shewed their connexion with the different ways of placing *different* numbers at the nodes of a Lattice, complete or incomplete, in such wise that the descending order of magnitude is in evidence in each row and in each column in the direction of the axes Ox, Oy of the lattice.

It was established that the lattice permutations of the assemblage

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_m^{p_m},$$

$p_1, p_2, \ldots$ being in descending order of magnitude, are enumerated by the number

$$\frac{(p_1+p_2+\ldots+p_{m-1}+p_m)!}{(p_1+m-1)!\,(p_2+m-2)!\ldots(p_{m-1}+1)!\,p_m!}\,\Pi\,(p_s-p_t-s+t),$$

where $s<t$ and the product Π has reference to every pair of numbers p_s, p_t that can be selected from the assemblage

$$p_1, p_2, \ldots p_m.$$

In the succeeding chapter the idea of the greater index of a permutation p was introduced and it was stated, but not proved, that the sum

$$\Sigma x^p,$$

extended to *every* permutation of the assemblage, has the expression

$$\frac{(1)\,(2)\ldots(\mathrm{p}_1+\mathrm{p}_2+\ldots+\mathrm{p}_\mathrm{m})}{(1)\,(2)\ldots(\mathrm{p}_1)\,.\,(1)\,(2)\ldots(\mathrm{p}_2)\ldots\ldots(1)\,(2)\ldots(\mathrm{p}_\mathrm{m})}.$$

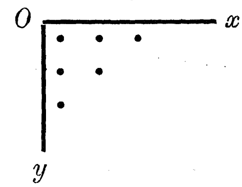
We may call this the Permutation Function. It will be established presently*. In the meantime we observe that we may form the sum Σx^p, extending it merely to every Lattice Permutation of the assemblage. We thus obtain a Lattice Permutation Function, or briefly a Lattice Function, which is of fundamental importance. From the manner of formation it is a

* *Post* Chapter IV.

finite and integral function of x with positive terms and integral coefficients. The coefficient of x^p denotes the number of lattice permutations which have the same index p and the sum of the coefficients must be equal to the number of lattice permutations.

The assemblage $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_m^{p_m}$ gives rise to the incomplete Lattice containing $p_1, p_2, \ldots p_m$ nodes in the successive rows as well as to the Lattice Permutations and the Lattice Function.

Consider the assemblage $\alpha^3\beta^2\gamma$ and with it the lattice



It was shewn that we can place the first six integers at its nodes so that there is descending order of magnitude in the directions Ox, Oy in just as many ways as the assemblage possesses lattice permutations.

Taking any such arrangement of numbers

$$\begin{array}{l}641\\53\\2\end{array}$$

we form the corresponding lattice permutation by commencing with α because 6 is in the first row, we follow with β because 5 is in the second row; then with α because 4 is in the first row and so on, and finally reach the permutation

$$\begin{array}{cccccc}\alpha&\beta&\alpha&\beta&\gamma&\alpha\\&2&&&5&\end{array}$$

Here the 2nd and 5th letters precede letters which are prior in alphabetical order, so that the greater index is $2+5=7$.

The complete calculation for the Lattice Function is as follows:

$$\begin{array}{cccc}
\begin{array}{l}654\\32\\1\end{array} & \begin{array}{l}654\\31\\2\end{array} & \begin{array}{l}653\\42\\1\end{array} & \begin{array}{l}653\\41\\2\end{array}\\
\alpha\alpha\alpha\beta\beta\gamma & \alpha\alpha\alpha\beta\gamma|\beta & \alpha\alpha\beta|\alpha\beta\gamma & \alpha\alpha\beta|\alpha\gamma|\beta\\
x^0 & x^5 & x^3 & x^8\\[1em]
\begin{array}{l}652\\43\\1\end{array} & \begin{array}{l}652\\41\\3\end{array} & \begin{array}{l}651\\43\\2\end{array} & \begin{array}{l}651\\42\\3\end{array}\\
\alpha\alpha\beta\beta|\alpha\gamma & \alpha\alpha\beta\gamma|\alpha\beta & \alpha\alpha\beta\beta\gamma|\alpha & \alpha\alpha\beta\gamma|\beta|\alpha\\
x^4 & x^4 & x^5 & x^9
\end{array}$$

631	632	641	642
52	51	52	51
4	4	3	3
$\alpha\beta\gamma\mid\alpha\beta\mid\alpha$	$\alpha\beta\gamma\mid\alpha\alpha\beta$	$\alpha\beta\mid\alpha\gamma\mid\beta\mid\alpha$	$\alpha\beta\mid\alpha\gamma\mid\alpha\beta$
x^8	x^3	x^{11}	x^6

641	643	642	643
53	51	53	52
2	2	1	1
$\alpha\beta\mid\alpha\beta\gamma\mid\alpha$	$\alpha\beta\mid\alpha\alpha\gamma\mid\beta$	$\alpha\beta\mid\alpha\beta\mid\alpha\gamma$	$\alpha\beta\mid\alpha\alpha\beta\gamma$
x^7	x^7	x^6	x^2

For a reason that will appear we write the Lattice Function $L(3, 2, 1; \infty)$ so that

$$L(3, 2, 1; \infty) = 1 + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + x^9 + x^{11}.$$

The contacts which contribute to the index have been indicated, above, by dividing lines. In the present instance we have either 0, 1, 2 or 3 dividing lines, and putting the terms which arise from the same number of lines together we may write

$$L(3, 2, 1; \infty) = 1 + (x^2 + 2x^3 + 2x^4 + 2x^5) + (2x^6 + 2x^7 + 2x^8 + x^9) + x^{11}.$$

When the lattice is complete with l columns and m rows we will write the lattice function formed in this manner

$$L(l; m; \infty);$$

when the lattice is incomplete with successive rows involving $p_1, p_2, \ldots p_m$ nodes we will write it

$$L(p_1 p_2 \ldots p_m; \infty).$$

437. It is a remarkable fact that the enumerating generating function of the plane partitions of the system $(l; m; \infty)$ has the expression

$$GF(l; m; \infty) = \frac{L(l; m; \infty)}{(1)(2)\ldots(lm)};$$

and here we can anticipate the generalization appertaining to the incomplete lattice; for denoting the enumerating generating function of the plane partitions, appertaining to successive rows involving $p_1, p_2, \ldots p_m$ nodes, by $GF(p_1 p_2 \ldots p_m; \infty)$, the part magnitude being unrestricted, it will be shewn that

$$GF(p_1 p_2 \ldots p_m; \infty) = \frac{L(p_1 p_2 \ldots p_m; \infty)}{(1)(2)\ldots(\Sigma p)}.$$

We establish these results before considering a restriction upon the part magnitude.

438. In the simplest case take α_1, α_2, β_1, β_2 numbers at the corners of a square and subject to descending orders along the rows and columns

$$\begin{matrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{matrix}$$

These numbers can be arranged collectively in two descending orders, viz. $\alpha_1, \alpha_2, \beta_1, \beta_2$, and $\alpha_1, \beta_1, \alpha_2, \beta_2$; and we may say that these numbers must satisfy one or other of the conditions

$$\alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \beta_2,$$
$$\alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \beta_2.$$

These conditions do not overlap, because we must have either $\alpha_2 \geqslant \beta_1$ *or* $\beta_1 > \alpha_2$. To obtain $\Sigma x^{\alpha_1+\alpha_2+\beta_1+\beta_2}$ for the first condition we put

$$\beta_1 = \beta_2 + A, \quad \alpha_2 = \beta_2 + A + B, \quad \alpha_1 = \beta_2 + A + B + C,$$

where A, B, C are arbitrary integers, zero not being excluded.

Hence $$\Sigma x^{\alpha_1+\alpha_2+\beta_1+\beta_2} = \Sigma x^{4\beta_2+3A+2B+C},$$

and since β_2, A, B, C may each assume any value from 0 to ∞ the result is

$$\frac{1}{(1)\,(2)\,(3)\,(4)}.$$

For the second condition we put

$$\alpha_2 = \beta_2 + A, \quad \beta_1 = \beta_2 + A + B + 1, \quad \alpha_1 = \beta_2 + A + B + C + 1,$$

and thence $$\Sigma x^{\alpha_1+\alpha_2+\beta_1+\beta_2} = \Sigma x^{4\beta_2+3A+2B+C+2}$$

$$= \frac{x^2}{(1)\,(2)\,(3)\,(4)}.$$

By addition we find that the enumerating function for the system $(2\,;\,2\,;\,\infty)$ is

$$\frac{1+x^2}{(1)\,(2)\,(3)\,(4)} \quad \text{or} \quad \frac{1}{(1)\,(2)^2\,(3)}.$$

When we compare the conditions

$$\alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \beta_2$$
$$\alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \beta_2$$

with the lattice permutations of the assemblage $\alpha\alpha\beta\beta$, viz.

$$\alpha\alpha\beta\beta$$
$$\alpha\beta\alpha\beta$$

we notice a connexion between them and observe also that each condition gives rise to a portion of the enumerating function

$$\frac{x^p}{(1)\,(2)\,(3)\,(4)},$$

where p is the greater index of the corresponding lattice permutation.

The whole enumerating function is

$$\frac{L(2\,;\,2\,;\,\infty)}{(1)\,(2)\,(3)\,(4)}.$$

439. Again consider a case which is not quite so simple, vi

$$\begin{matrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{matrix}$$

a plane partition of the system $(2\,;\,3\,;\,\infty)$.

Here the numbers may be in five descending orders when taken collectively, viz.

$$\begin{matrix} \alpha_1, & \alpha_2, & \beta_1, & \beta_2, & \gamma_1, & \gamma_2, \\ \alpha_1, & \alpha_2, & \beta_1, & \gamma_1, & \beta_2, & \gamma_2, \\ \alpha_1, & \beta_1, & \alpha_2, & \beta_2, & \gamma_1, & \gamma_2, \\ \alpha_1, & \beta_1, & \alpha_2, & \gamma_1, & \beta_2, & \gamma_2, \\ \alpha_1, & \beta_1, & \gamma_1, & \alpha_2, & \beta_2, & \gamma_2, \end{matrix}$$

corresponding to the five lattice permutations of the assemblage $\alpha^2\beta^2\gamma^2$, viz.

$$\begin{gathered} \alpha\alpha\beta\beta\gamma\gamma, \\ \alpha\alpha\beta\gamma|\beta\gamma, \\ \alpha\beta|\alpha\beta\gamma\gamma, \\ \alpha\beta|\alpha\gamma|\beta\gamma, \\ \alpha\beta\gamma|\alpha\beta\gamma, \end{gathered}$$

and we may say that the numbers must satisfy one or other of the five sets of Diophantine inequalities

$$\begin{gathered} \alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \beta_2 \geqslant \gamma_1 \geqslant \gamma_2, \\ \alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \gamma_1 > \beta_2 \geqslant \gamma_2, \\ \alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \beta_2 \geqslant \gamma_1 \geqslant \gamma_2, \\ \alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \gamma_1 > \beta_2 \geqslant \gamma_2, \\ \alpha_1 \geqslant \beta_1 \geqslant \gamma_1 > \alpha_2 \geqslant \beta_2 \geqslant \gamma_2, \end{gathered}$$

conditions which do not overlap.

Comparing them with the lattice permutations which are lined, as a preparation to finding the index, we notice the correspondence between the places where the symbol $>$ occurs, and the lines in the permutations.

If we now take any one of these sets of inequalities we find that it gives rise to a portion $x^p/(1)\,(2)\,\ldots\,(6)$ of the enumerating function, where p is the index of the corresponding permutation. To see how this is observe that if the symbol $>$ occurs between the sth and $(s+1)$th letters of the set of inequalities, we put

$$s\text{th letter} = (s+1)\text{th letter} + N + 1$$

to find the corresponding portion of the enumerating function, and this unit necessarily persists in the expressions of each of the first $s-1$ letters of the inequalities. When these come to be added together as we added them in the case of the system $(2; 2; \infty)$ it is clear that the integer s will be outstanding in the exponent of x in the numerator of the function. The occurrence of this particular symbol $>$ after the sth letter thus contributes s to the exponent referred to. On the whole the symbols $>$ in the set of inequalities will contribute a sum of numbers to the exponent of x in the numerator; and this sum of numbers we have defined to be p the greater index of the corresponding permutation.

Hence taking all the sets of inequalities the enumerating function is seen to be

$$\frac{\Sigma x^p}{(1)(2)\dots(6)},$$

or

$$\frac{L(2; 3; \infty)}{(1)(2)\dots(6)}.$$

Looking at the lattice permutations as lined, it is seen at once that the lattice function is

$$1+(x^2+x^3+x^4)+x^6,$$

so that

$$GF(2; 3; \infty)=\frac{1+x^2+x^3+x^4+x^6}{(1)(2)\dots(6)}=\frac{1}{(1)(2)^2(3)^2(4)},$$

since

$$1+x^2+x^3+x^4+x^6=\frac{(5)(6)}{(2)(3)}.$$

The reasoning is quite general and establishes that

$$GF(l; m; \infty)=\frac{L(l; m; \infty)}{(1)(2)\dots(lm)},$$

a very interesting result.

Precisely the same argument applies to the incomplete lattice, so that also

$$GF(p_1p_2p_3\dots p_m; \infty)=\frac{L(p_1p_2p_3\dots p_m; \infty)}{(1)(2)\dots(\Sigma p)},$$

establishing beyond doubt that the method of investigation includes in its purview the subject of plane partitions at the nodes of any incomplete lattice.

440. In obtaining the lattice function the index of a permutation arises as the sum of s numbers when s has values from zero to μ where μ is a number to be determined. To use a phrase employed elsewhere in this work, such a permutation involves s major contacts. If we restrict attention to these permutations we arrive at what is termed a

Sub-Lattice Function of Order s.

This is written

$$L_s(l; m; \infty) \text{ or } L_s(p_1p_2\ldots p_m; \infty),$$

so that

$$L(l; m; \infty) = L_0(l; m; \infty) + L_1(l; m; \infty) + L_2(l; m; \infty) + \ldots + L_\mu(l; m; \infty).$$

In particular we see that $L_0(l; m; \infty)$ or $L_0(p_1p_2\ldots p_m; \infty)$ is always unity. Thus we have shewn that

$$L_0(3; 2; \infty) = 1, \quad L_1(3; 2; \infty) = x^2 + x^3 + x^4, \quad L_2(3; 2; \infty) = x^6.$$

The sub-lattice functions possess elegant properties. One of the most elegant is that which enables us to define a new lattice function

$$L(l; m; n) \text{ or } L(p_1p_2\ldots p_m; n)$$

appertaining to plane partitions in which the part magnitude is limited by the number n. The definition of $L(p_1p_2\ldots p_m; n)$ is given by the relation

$$\begin{aligned} L(p_1p_2\ldots p_m; n) = {} & (\mathbf{n}+\mathbf{1})(\mathbf{n}+\mathbf{2})\ldots(\mathbf{n}+\mathbf{\Sigma p})\, L_0(p_1p_2\ldots p_m; \infty) \\ & + (\mathbf{n})(\mathbf{n}+\mathbf{1})\ldots(\mathbf{n}+\mathbf{\Sigma p}-\mathbf{1})\, L_1(p_1p_2\ldots p_m; \infty) \\ & + (\mathbf{n}-\mathbf{1})(\mathbf{n})\ldots(\mathbf{n}+\mathbf{\Sigma p}-\mathbf{2})\, L_2(p_1p_2\ldots p_m; \infty) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + (\mathbf{n}-\boldsymbol{\mu}+\mathbf{1})(\mathbf{n}-\boldsymbol{\mu}+\mathbf{2})\ldots(\mathbf{n}-\boldsymbol{\mu}+\mathbf{\Sigma p})\, L_\mu(p_1p_2\ldots p_m; \infty). \end{aligned}$$

The new sub-lattice function is

$$L_s(p_1p_2\ldots p_m; n) = (\mathbf{n}-\mathbf{s}+\mathbf{1})(\mathbf{n}-\mathbf{s}+\mathbf{2})\ldots(\mathbf{n}-\mathbf{s}+\mathbf{\Sigma p})\, L_s(p_1p_2\ldots p_m; \infty).$$

We are now ready to investigate the expressions of

$$GF(l; m; n) \text{ and } GF(p_1p_2\ldots p_m; n).$$

We have to modify the previous work so as to take account of the circumstance that the part magnitude is now restricted not to exceed n.

441. Recalling the simple case of the plane partition

$$\begin{matrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{matrix}$$

we had the five distinct parts of the summation:

(i) $\alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \beta_2 \geqslant \gamma_1 \geqslant \gamma_2$ giving $\Sigma x^{6\gamma_2+5A+4B+3C+2D+E}$,

(ii) $\alpha_1 \geqslant \alpha_2 \geqslant \beta_1 \geqslant \gamma_1 > \beta_2 \geqslant \gamma_2$,, $\Sigma x^{6\gamma_2+5A+4B+3C+2D+E+4}$,

(iii) $\alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \beta_2 \geqslant \gamma_1 \geqslant \gamma_2$,, $\Sigma x^{6\gamma_2+5A+4B+3C+2D+E+2}$,

(iv) $\alpha_1 \geqslant \beta_1 > \alpha_2 \geqslant \gamma_1 > \beta_2 \geqslant \gamma_2$,, $\Sigma x^{6\gamma_2+5A+4B+3C+2D+E+6}$,

(v) $\alpha_1 \geqslant \beta_1 \geqslant \gamma_1 > \alpha_2 \geqslant \beta_2 \geqslant \gamma_2$,, $\Sigma x^{6\gamma_2+5A+4B+3C+2D+E+3}$.

In dealing with the set of inequalities (i) we put

$$\gamma_1 = \gamma_2 + A, \quad \beta_2 = \gamma_2 + A + B, \quad \beta_1 = \gamma_2 + A + B + C, \quad \alpha_2 = \gamma_2 + A + B + C + D,$$
$$\alpha_1 = \gamma_2 + A + B + C + D + E,$$

and since α_1 cannot exceed n, so also $\gamma_2 + A + B + C + D + E$ cannot exceed n. We have therefore

$$\sum x^{6\gamma_2 + 5A + 4B + 3C + 2D + E},$$

where γ_2, A, B, C, D, E may assume any integer values, including zero, subject to the condition that $\gamma_2 + A + B + C + D + E \not> n$.

This is equivalent to enumerating line partitions composed of n or fewer parts, no part exceeding 6 in magnitude. The set of inequalities (i) therefore leads by a known theorem in line partitions to

$$\frac{(n+1)\,(n+2)\ldots(n+6)}{(1)\,(2)\ldots(6)}.$$

For the set of inequalities (ii) similarly we find

$$\alpha_1 = \gamma_2 + A + B + C + D + E + 1,$$

so that $\gamma_2 + A + B + C + D + E$ cannot exceed $n-1$ in magnitude and the sum $\sum x^{6\gamma_2 + 5A + 4B + 3C + 2D + E + 4}$ is clearly

$$\frac{(n)\,(n+1)\ldots(n+5)}{(1)\,(2)\ldots(6)}\,x^4.$$

So also in the case of each set of inequalities that is associated with the sub-lattice function of order 1

$$\gamma_2 + A + B + C + D + E$$

cannot exceed $n-1$ in magnitude.

Therefore sets (ii), (iii) and (v) contribute altogether

$$\frac{(n)\,(n+1)\ldots(n+5)}{(1)\,(2)\ldots(6)}\,L_1(2\,;3\,;\infty).$$

The remaining set (iv) involves putting

$$\alpha_1 = \gamma_2 + A + B + C + D + E + 2,$$

because the set is associated with the sub-lattice function of order 2, and this leads us to the final portion of the enumerating function

$$\frac{(n-1)\,(n)\ldots(n+4)}{(1)\,(2)\ldots(6)}\,L_2(2\,;3\,;\infty),$$

$L_2(2\,;3\,;\infty)$ being here $x^{2+4} = x^6$.

Hence $$GF(2\,;3\,;n)$$

$$= \frac{(n+1)\ldots(n+6)\,L_0(2;3;\infty) + (n)\ldots(n+5)\,L_1(2;3;\infty) + (n-1)\ldots(n+4)\,L_2(2;3;\infty)}{(1)\,(2)\ldots(6)}$$

$$= \frac{L(2;3;n)}{(1)\,(2)\ldots(6)}.$$

442. The reasoning is quite general and establishes the results

$$GF(l;m;n) = \frac{L(l;m;n)}{(1)(2)\dots(lm)},$$

$$GF(p_1p_2\dots p_m;n) = \frac{L(p_1p_2\dots p_m;n)}{(1)(2)\dots(\Sigma p)}.$$

443. From the result

$$GF(p_1p_2\dots p_m;n)$$

$$= \frac{(n+1)(n+2)\dots(n+6)}{(1)(2)\dots(\Sigma p)} L_0(p_1p_2\dots p_m;\infty)$$

$$+ \frac{(n)(n+1)\dots(n+5)}{(1)(2)\dots(\Sigma p)} L_1(p_1p_2\dots p_m;\infty)$$

$$+ \dots\dots\dots\dots\dots\dots$$

$$+ \frac{(n-\mu+1)(n-\mu+2)\dots(n-\mu+\Sigma p)}{(1)(2)\dots(\Sigma p)} L_\mu(p_1p_2\dots p_m;\infty)$$

we can immediately derive a valuable theorem.

Multiply each side by g^n and then sum each side from $n=0$ to $n=\infty$. The term

$$\sum_0^\infty g^n \frac{(n-s+1)(n-s+2)\dots(n-s+\Sigma p)}{(1)(2)\dots(\Sigma p)} L_s(p_1p_2\dots p_m;\infty),$$

which thus arises on the right-hand side, is equivalent to

$$\sum_s^\infty g^n \frac{(n-s+1)(n-s+2)\dots(n-s+\Sigma p)}{(1)(2)\dots(\Sigma p)} L_s(p_1p_2\dots p_m;\infty),$$

or to

$$g^s L_s(p_1p_2\dots p_m;\infty)\left\{1 + \frac{(\Sigma p+1)}{(1)} g + \frac{(\Sigma p+1)(\Sigma p+2)}{(1)(2)} g^2 + \frac{(\Sigma p+1)(\Sigma p+2)(\Sigma p+3)}{(1)(2)(3)} g^3 + \dots\right\}$$

$$= g^s L_s(p_1p_2\dots p_m;\infty)\frac{1}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^{\Sigma p})}$$

by a well-known summation in connexion with line partitions.

Hence $$\sum_0^\infty GF(p_1p_2\dots p_m;n)\,.\,g^n$$

$$= \frac{L_0(p_1p_2\dots p_m;\infty) + gL_1(p_1p_2\dots p_m;\infty) + \dots + g^\mu L_\mu(p_1p_2\dots p_m;\infty)}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^{\Sigma p})},$$

a noteworthy result which is a generalization of the well-known formula in line partitions

$$\frac{1}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^m)}$$
$$=\sum_0^\infty g^n \frac{(1)\,(2)\dots(\mathrm{m}+\mathrm{n})}{(1)\,(2)\dots(\mathrm{m})\,.\,(1)\,(2)\dots(\mathrm{n})}$$
$$=\Sigma\, g^n\, GF\,(m\,;\,1\,;\,n)^*.$$

It is thus noticeable that the sub-lattice functions play an important rôle in the Theory of Plane Partitions.

444. It is convenient, when no confusion can arise, to write

$$L_s\,(p_1 p_2 \dots p_m\,;\,\infty\,) = L_s.$$

Then writing x for g we obtain the result

$$\frac{L_0 + xL_1 + x^2L_2 + \dots + x^\mu L_\mu}{(1)\,(2)\dots(\Sigma\mathrm{p}+1)} = \sum_0^\infty x^n\, GF\,(p_1p_2\dots p_m\,;\,n),$$

the generalization of the formula

$$\frac{1}{(1)\,(2)\dots(\mathrm{m}+1)} = \sum_0^\infty x^n \frac{(1)\,(2)\dots(\mathrm{m}+\mathrm{n})}{(1)\,(2)\dots(\mathrm{m})\,.\,(1)\,(2)\dots(\mathrm{n})}\,.$$

445. We now make a further study of the sub-lattice functions and the first step is to find the value of μ, the highest order of sub-lattice function in the case of a complete lattice.

For a lattice of m rows and l columns form the rectangular scheme

$$\begin{array}{cccc}
\alpha_1 & \alpha_1 & \alpha_1 \dots & \alpha_1 \\
\alpha_2 & \alpha_2 & \alpha_2 \dots & \alpha_2 \\
\alpha_3 & \alpha_3 & \alpha_3 \dots & \alpha_3 \\
\dots & \dots & \dots & \dots \\
\alpha_m & \alpha_m & \alpha_m \dots & \alpha_m
\end{array}$$

where there are l columns.

Reading this parallel to the arrow commencing at the origin we obtain the lattice permutation

$$\alpha_1 \quad \alpha_2|\alpha_1 \quad \alpha_3|\alpha_2|\alpha_1 \dots \alpha_m|\alpha_{m-1}|\alpha_{m-2} \quad \alpha_m|\alpha_{m-1} \quad \alpha_m.$$

This is the permutation which involves the maximum number of major contacts and corresponds to the sub-lattice function of highest order. There is only one such permutation, so that the associated sub-lattice function is a single power of x.

* See *ante* Art. 343.

We may without loss of generality suppose $l \geqslant m$ and then the number of major contacts is

$$1+2+3+\ldots+m-2+(l-m+1)(m-1)+(m-2)+(m-3)+\ldots+3+2+1$$
$$=(l-1)(m-1).$$

Hence for the complete lattice the maximum value of μ is $(l-1)(m-1)$.

The same method is applicable in the case of the incomplete lattice, but no convenient expression for μ is apparently obtainable.

446. The next step is to determine the expression of

$$L_{(l-1)(m-1)}(l\,;\,m\,;\,\infty)$$

as a power of x.

The dividing lines which mark the major contacts of the permutation occur

(i) In $m-2$ groups involving $1, 2, \ldots m-2$ lines respectively.

(ii) In $l-m+1$ groups involving each $m-1$ lines.

(iii) In $m-2$ groups involving $m-2, m-3, \ldots 2, 1$ lines respectively.

Let the sought exponent of x be $\pi_1+\pi_2+\pi_3$; π_1, π_2, π_3 corresponding respectively to (i), (ii) and (iii).

Then

$$\pi_1 = 2+(4+5)+(7+8+9)+\ldots+\{\tfrac{1}{2}(m^2-3m+4)+\ldots+\tfrac{1}{2}(m^2-m-2)\}$$
$$=\tfrac{1}{2}(1.2^2+2.3^2+3.4^2+\ldots \text{ to } m-2 \text{ terms})$$
$$=\tfrac{1}{24}m(m-1)(m-2)(3m-1),$$

$$\pi_2=\tfrac{1}{2}(m-1)m^2+\tfrac{1}{2}(m-1)m(m+2)+\tfrac{1}{2}(m-1)m(m+4)+\ldots \text{ to } l-m+1 \text{ terms}$$
$$=\tfrac{1}{2}(m-1)lm(l-m+1),$$

$$\pi_3=(lm-2)+(lm-4+lm-5)+(lm-7+lm-8+lm-9)+\ldots \text{ to } m-2 \text{ terms}$$
$$=\tfrac{1}{2}lm(m-1)(m-2)-\tfrac{1}{24}m(m-1)(m-2)(3m-1),$$

whence $$\pi_1+\pi_2+\pi_3=\tfrac{1}{2}l(l-1)m(m-1),$$

and $$L_{(l-1)(m-1)}(l\,;\,m\,;\,\infty)=x^{\frac{1}{2}l(l-1)m(m-1)}.$$

It will be noticed that the exponent of x may be written

$$\tfrac{1}{2}lm\mu,$$

or $$lm\mu-\tfrac{1}{2}lm\mu.$$

447. We next establish that in L_s for every term x^p there is a corresponding term x^{lms-p}. These two terms, as shewn above, coalesce when $s=\mu$.

If, in the permutation

$$\alpha_1 \quad \alpha_2|\alpha_1 \quad \alpha_3|\alpha_2|\alpha_1 \ldots \alpha_m|\alpha_{m-1}|\alpha_{m-2} \quad \alpha_m|\alpha_{m-1} \quad \alpha_m$$

considered above, we fix upon a dividing line and arrange the letters to the right of it in alphabetical order (or in ascending order of suffix), thus obliterating the lines to the right of the one fixed upon, we obtain a lattice permutation involving (suppose) s lines which yields x to the lowest power that occurs in L_s. In fact we thus obtain a maximum condensation of s lines towards the origin of the permutation. If on the other hand we fix upon a dividing line which is the sth from the end of the permutation and arrange the letters to the left of it in alphabetical order, we obtain a lattice permutation involving s lines which yields x to the highest power that occurs in L_s. In fact we thus obtain a maximum condensation of s lines towards the end of the permutation.

When the lattice is complete we may in any derived lattice permutation write α_{m-s+1} for α_s and invert the order of the whole of the letters. We thus obtain another lattice permutation, and the latter has the same number of major contacts as the former and thus contributes to the same sub-lattice function. For a succession

$$\alpha_p|\alpha_q \qquad (p > q)$$

in the former, α_p being the kth letter from the left of the permutation, becomes

$$\alpha_{m-q+1}|\alpha_{m-p+1} \qquad (m-q+1 > m-p+1)$$

in the latter, α_{m-q+1} being the $(lm-k)$th letter from the left of the permutation.

Hence if in the former the components of the index are

$$k_1,\ k_2, \ldots k_s,$$

in the latter the components of the index are

$$lm - k_1,\quad lm - k_2, \ldots lm - k_s.$$

If the index of the former be p, that of the latter must be

$$lms - p.$$

It is therefore proved that in $L_s(l;\, m;\, \infty)$ there are corresponding terms

$$x^p \text{ and } x^{lms-p},$$

and thence we conclude that L_s is centrically symmetrical both as regards the powers of x and the coefficients. In fact

$$x^p \text{ and } x^{lms-p}$$

occur with the same coefficients.

If ϵ be the lowest power of x in L_s determined as above directed, the highest power will be $lms - \epsilon$.

Ex. gr. For the system $(3;\ 3;\ \infty)$

$$L_0 = 1,$$
$$L_1 = x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7,$$
$$L_2 = 2(x^6 + x^7 + 2x^8 + 2x^9 + 2x^{10} + x^{11} + x^{12}),$$
$$L_3 = x^{11} + 2x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + x^{16},$$
$$L_4 = x^{18},$$

verifying the above relations.

The permutation connected with L_4 is

$$\alpha\beta|\alpha\gamma|\beta|\alpha\gamma|\beta\gamma$$
$$2\ \ 4\ \ 5\ \ 7$$

where the components of the index are entered.

It will be noted that 2, $2+4$, $2+4+5$, $2+4+5+7$ give the lowest powers of x that occur in L_1, L_2, L_3, L_4, respectively and that the highest powers are given by 7, $7+5$, $7+5+4$, $7+5+4+2$; in conformity with the above theory.

The result of writing $\frac{1}{x}$ for x in L_s is to convert it into $x^{-lms}L_s$.

448. If e_s be the least exponent of x that occurs in L_s we can find a relation between e_s and $e_{\mu-s}$.

In the permutation connected with L_μ, viz.

$$\alpha_1\quad \alpha_2|\alpha_1\quad \alpha_3|\alpha_2|\alpha_1 \ldots \alpha_m|\alpha_{m-1}|\alpha_{m-2}\quad \alpha_m|\alpha_{m-1}\quad \alpha_m,$$

e_s is equal to the sum of the components due to the first s major contacts—say

$$e_s = k_1 + k_2 + \ldots + k_s,$$

$e_{\mu-s}$ is equal to the sum of the whole of the components minus the sum of the last s components. Now the last s components are

$$lm - k_s,\quad lm - k_{s-1},\ \ldots\ lm - k_1,$$

therefore
$$e_{\mu-s} = e_\mu - lms + e_s,$$

or
$$e_{\mu-s} - e_s = \tfrac{1}{2}lm\mu - lms = \tfrac{1}{2}lm(\mu - 2s).$$

Moreover if f_s be the highest exponent of x that occurs in L_s, the symmetry of the permutation shews that

$$f_{\mu-s} - f_s = \tfrac{1}{2}lm(\mu - 2s),$$

and we deduce that

$$x^{e_{\mu-s}} + x^{f_{\mu-s}} = x^{\frac{1}{2}lm(\mu-2s)}(x^{e_s} + x^{f_s}),$$

a result which foreshadows the theorem

$$L_{\mu-s} = x^{\frac{1}{2}lm(\mu-2s)}L_s,$$

which will be presently established.

Combining the above with the relation

$$e_s + f_s = lms,$$

we find
$$e_s + f_{\mu-s} = \tfrac{1}{2} lm\mu,$$

so that $e_s + f_{\mu-s}$ does not vary with s.

449. To prove that $L_{\mu-s}$ and L_s only differ by a factor which is a power of x we suppose that x^p is a term of L_s derived from *any* s of the lines appertaining to the permutation connected with L_μ; the remaining $\mu - s$ lines can be obliterated by arranging the letters in alphabetical order between the chosen lines. Thus

$$p = r_1 + r_2 + \ldots + r_s \text{ suppose.}$$

If instead we retain *only* those lines which we have just obliterated we obtain a term in $L_{\mu-s}$ in which the exponent of x is $\frac{1}{2}lm\mu - p$ because the sum of the components due to the whole of the lines is $\frac{1}{2}lm\mu$. Hence if $L_s = \Sigma x^p$, we have

$$L_{\mu-s} = \Sigma x^{\frac{1}{2}lm\mu - p}.$$

Now it has been shewn (Art. 447) that if $L_s = \Sigma x^p$, it follows that

$$\Sigma x^{-p} = x^{-lms} \Sigma x^p,$$

so that

$$L_{\mu-s} = x^{\frac{1}{2}lm\mu} \Sigma x^{-p} = x^{\frac{1}{2}lm(\mu-2s)} \Sigma x^p = x^{\frac{1}{2}lm(\mu-2s)} L_s.$$

450. If a lattice be read by columns instead of by rows its specification changes from a certain partition to its conjugate. It is a trivial remark that the generating function which enumerates the associated plane partitions is not altered by the change. That is to say that if

$$p_1 p_2 \ldots, \quad q_1 q_2 \ldots$$

be conjugate unipartite partitions

$$GF(p_1 p_2 \ldots;\ n) = GF(q_1 q_2 \ldots;\ n),$$

a relation which we can express in the form

$$\frac{L(p_1 p_2 \ldots;\ n)}{(1)(2)\ldots(\Sigma p)} = \frac{L(q_1 q_2 \ldots;\ n)}{(1)(2)\ldots(\Sigma q)},$$

wherein $\Sigma p = \Sigma q$. Hence

$$L(p_1 p_2 \ldots;\ n) = L(q_1 q_2 \ldots;\ n),$$

establishing that changing the lattice into its conjugate does not alter the lattice function for any value of n.

Moreover the last relation may be written

$$\begin{aligned} &(n+1)(n+2)\ldots(n+\Sigma p)\, L_0(p_1 p_2 \ldots;\ \infty) \\ &+ (n)(n+1)\ldots(n+\Sigma p - 1)\, L_1(p_1 p_2 \ldots;\ \infty) + \ldots \\ &= (n+1)(n+2)\ldots(n+\Sigma q)\, L_0(q_1 q_2 \ldots;\ \infty) \\ &+ (n)(n+1)\ldots(n+\Sigma q - 1)\, L_1(q_1 q_2 \ldots;\ \infty) + \ldots. \end{aligned}$$

Putting herein $n = 1, 2, \ldots$ in succession we establish that

$$L_s(p_1p_2\ldots;\ \infty) = L_s(q_1q_2\ldots;\ \infty),$$

and thence
$$L_s(p_1p_2\ldots;\ n) = L_s(q_1q_2\ldots;\ n),$$

proving that the sub-lattice functions do not alter when we pass from a lattice to its conjugate.

451. The fundamental relation

$$GF(l;\ m;\ n) = \frac{(\mathbf{n+1})(\mathbf{n+2})\ldots(\mathbf{n+lm})L_0 + (\mathbf{n})(\mathbf{n+1})\ldots(\mathbf{n+lm-1})L_1 + \ldots}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{lm})}$$

exhibits $GF(l;\ m;\ n)$ as a linear function of the sub-lattice functions. By giving n the special values $0, 1, 2, \ldots$ in succession we obtain the relations

$$GF(l;\ m;\ 0) = L_0(l;\ m;\ \infty) = 1,$$

$$GF(l;\ m;\ 1) = \frac{(\mathbf{lm+1})}{(\mathbf{1})} + L_1(l;\ m;\ \infty),$$

$$GF(l;\ m;\ 2) = \frac{(\mathbf{lm+1})(\mathbf{lm+2})}{(\mathbf{1})(\mathbf{2})} + \frac{(\mathbf{lm+1})}{(\mathbf{1})}L_1 + L_2,$$

....

$$GF(l;\ m;\ \mu) = \frac{(\mathbf{lm+1})\ldots(\mathbf{lm+\mu})}{(\mathbf{1})\ldots(\boldsymbol{\mu})} + \frac{(\mathbf{lm+1})\ldots(\mathbf{lm+\mu-1})}{(\mathbf{1})\ldots(\boldsymbol{\mu-1})}L_1 + \ldots + L_\mu,$$

$$GF(l;\ m;\ \mu+s) = \frac{(\mathbf{lm+1})\ldots(\mathbf{lm+\mu+s})}{(\mathbf{1})(\mathbf{2})\ldots(\boldsymbol{\mu}+\mathbf{s})} + \frac{(\mathbf{lm+1})\ldots(\mathbf{lm+\mu+s-1})}{(\mathbf{1})(\mathbf{2})\ldots(\boldsymbol{\mu}+\mathbf{s-1})}L_1 + \ldots$$
$$+ \frac{(\mathbf{lm+1})\ldots(\mathbf{lm+s})}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{s})}L_\mu,$$

and thence

$$L_0 = 1,$$

$$L_1 = GF(l;\ m;\ 1) - \frac{(\mathbf{lm+1})}{(\mathbf{1})},$$

$$L_2 = GF(l;\ m;\ 2) - \frac{(\mathbf{lm+1})}{(\mathbf{1})}GF(l;\ m;\ 1) + x\frac{(\mathbf{lm})(\mathbf{lm+1})}{(\mathbf{1})(\mathbf{2})},$$

....

$$L_\mu = GF(l;\ m;\ \mu) - \frac{(\mathbf{lm+1})}{(\mathbf{1})}GF(l;\ m;\ \mu-1)$$
$$+ x\frac{(\mathbf{lm})(\mathbf{lm+1})}{(\mathbf{1})(\mathbf{2})}GF(l;\ m;\ \mu-2) - \ldots$$
$$+ (-)^k x^{\frac{1}{2}(k-1)k}\frac{(\mathbf{lm-k+2})\ldots(\mathbf{lm+1})}{(\mathbf{1})\ldots(\mathbf{k})}GF(l;\ m;\ \mu-k) + \ldots$$
$$+ (-)^\mu x^{\frac{1}{2}(\mu-1)\mu}\frac{(\mathbf{lm}-\boldsymbol{\mu}+\mathbf{2})\ldots(\mathbf{lm+1})}{(\mathbf{1})\ldots(\boldsymbol{\mu})}.$$

This series may be continued, the left-hand side becoming zero.

We then get a number of difference equations satisfied by the function $GF(l; m; n)$.

From the first series we at once find the general expression of $L_1(l; m; \infty)$, viz.

$$L_1(l; m; \infty) = \frac{(l+1)(l+2)\ldots(l+m)}{(1)(2)\ldots(m)} - \frac{(lm+1)}{(1)},$$

because we know the expression of the line-partition generating function $GF(l; m; 1)$.

452. Another method of investigation leads to another expression for L_1.

Taking the assemblage of letters

$$\alpha_1^l \alpha_2^l \ldots \alpha_m^l,$$

and observing that the index of the lattice permutation with which we have to deal has only one component, let the index arise from the conjunction $\alpha_v \alpha_u$, where $v > u$. The whole permutation will be

$$A)\, \alpha_v | \alpha_u \,(B,$$

where in the space A there is any succession of letters in ascending order of subscripts to v and in the space B any succession such that the subscripts are in ascending order from u.

The least index is obtained when the permutation is

$$\alpha_1^l \alpha_2^l \ldots \alpha_{u-1}^l \alpha_u \alpha_{u+1} \ldots \alpha_{v-1})\, \alpha_v | \alpha_u \,(\alpha_u^{l-2} \alpha_{u+1}^{l-1} \ldots \alpha_{v-1}^{l-1} \alpha_v^{l-1} \alpha_{v+1}^l \ldots \alpha_m^l,$$

because no one of the letters $\alpha_1, \alpha_2, \ldots \alpha_{u-1}$ can be in the space B without introducing another dividing line and the letters $\alpha_u, \alpha_{u+1}, \ldots \alpha_{v-1}$ must each occur at least once in the space A to satisfy the lattice condition.

This gives the index $v + (l-1)u - l + 1$, and the term

$$x^{v+(l-1)u-l+1}.$$

The greatest index is derived from the permutation

$$\alpha_1^l \alpha_2^l \ldots \alpha_{u-1}^l \alpha_u^{l-1} \alpha_{u+1}^{l-1} \ldots \alpha_{v-1}^{l-1} \alpha_v^{l-2})\, \alpha_v | \alpha_u \,(\alpha_{u+1} \alpha_{u+2} \ldots \alpha_{v-1} \alpha_v \alpha_{v+1}^l \ldots \alpha_m^l,$$

because no one of the letters $\alpha_{v+1}, \alpha_{v+2}, \ldots \alpha_m$ can be in the space A without introducing another dividing line and the letters $\alpha_{u+1}, \alpha_{u+2}, \ldots \alpha_v$ must each occur at least once in the space B to satisfy the lattice condition.

This gives the index $(l-1)v + u - 1$ and the term

$$x^{(l-1)v+u-1}.$$

In regard to the space A and the letter α_v, to the immediate left of the dividing line, we have to determine all the successions ranging from

$$\alpha_u \alpha_{u+1} \ldots \alpha_{v-1} \alpha_v \quad \text{to} \quad \alpha_u^{l-1} \alpha_{u+1}^{l-1} \ldots \alpha_{v-1}^{l-1} \alpha_v^{l-2},$$

that can be placed between the succession

$$\alpha_1^l \alpha_2^l \ldots \alpha_{u-1}^l$$

and the dividing line, in order to form a permutation of the desired nature.

Suppose that one such succession is

$$\alpha_u^{t_1}\alpha_{u+1}^{t_2}\ldots\alpha_v^{t_{v-u+1}};$$

$t_1, t_2, \ldots t_{v-u+1}$ may be any succession of $v-u+1$ numbers in descending order of magnitude, such that no number is greater than $l-1$ or less than unity. Or, subtracting unity from each number we may say that the succession

$$t_1-1,\ t_2-1, \ldots t_{v-u+1}-1$$

is a partition at the points of a one row lattice of $v-u+1$ nodes such that no part exceeds $l-2$ in magnitude.

The generating function enumerating such partitions is

$$\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{v}-\mathrm{u}-1)}{(1)(2)\ldots(\mathrm{v}-\mathrm{u}+1)}.$$

Denote by $L_{1,vu}$ that portion of L_1 which arises from the major contact $\alpha_v\alpha_u$. Then

$$L_{1,vu}=x^{v+(l-1)u-l+1}\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{v}-\mathrm{u}-1)}{(1)(2)\ldots(\mathrm{v}-\mathrm{u}+1)},$$

leading to

$$L_1(l;m;\infty)=\sum_v\sum_u x^{v+(l-1)u-l+1}\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{v}-\mathrm{u}-1)}{(1)(2)\ldots(\mathrm{v}-\mathrm{u}+1)}.$$

Put herein $v=u+j$; then for a constant value of j we have

$$L_1(l;m;\infty)_j=\{x^{j+1}+x^{j+l+1}+\ldots+x^{lm-(l-1)(j+1)}\}\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{j}-1)}{(1)(2)\ldots(\mathrm{j}+1)}$$

$$=x^{j+1}\frac{(\mathfrak{l}\mathrm{m}-\mathfrak{l}\mathrm{j})}{(\mathfrak{l})}\cdot\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{j}-1)}{(1)(2)\ldots(\mathrm{j}+1)},$$

and consequently giving j all values from 1 to $m-1$,

$$L_1(l;m;\infty)=x^2\frac{(\mathfrak{l}\mathrm{m}-\mathfrak{l})}{(\mathfrak{l})}\cdot\frac{(\mathfrak{l}-1)(\mathfrak{l})}{(1)(2)}$$

$$+x^3\frac{(\mathfrak{l}\mathrm{m}-2\mathfrak{l})}{(\mathfrak{l})}\cdot\frac{(\mathfrak{l}-1)(\mathfrak{l})(\mathfrak{l}+1)}{(1)(2)(3)}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+x^m\frac{(\mathfrak{l})}{(\mathfrak{l})}\cdot\frac{(\mathfrak{l}-1)(\mathfrak{l})\ldots(\mathfrak{l}+\mathrm{m}-2)}{(1)(2)\ldots(\mathrm{m})};$$

a result which, as not depending upon the enumerating function $GF(l;m;\infty)$, is more interesting than the one reached in the previous Article.

CHAPTER IV

EXCURSUS ON PERMUTATION FUNCTIONS

453. If we had dealt with the whole of the permutations instead of merely with the lattice permutations, we should have similarly arrived at an enumerating function

$$\frac{PF\,(p_1 p_2 p_3 \ldots p_m;\ \infty\,)}{(1)\,(2)\,(3)\ldots(\Sigma p)},$$

but what arrangements should we have enumerated?

Taking the last particular case examined above (Art. 441) we should have written down the 90 permutations of $\alpha\alpha\beta\beta\gamma\gamma$ and the corresponding 90 sets of Diophantine inequalities in which α_1, β_1, γ_1 would always precede α_2, β_2, γ_2 respectively, but there would be no further restriction of order between the letters α_1, α_2, β_1, β_2, γ_1, γ_2. Consequently in the plane partitions we should *retain the descending order of magnitude in the rows while abolishing it in the columns.* We are thus in the general case dealing with arrangements at the nodes of a lattice, complete or incomplete, the only condition being that a descending order of magnitude is to be in evidence *in each row.* Now in a row containing p_s nodes we can place any line partition of any number so long as the number of the parts does not exceed p_s. The generating function for these is

$$\frac{1}{(1)\,(2)\ldots(p_s)};$$

and, regarding all the rows, we find that the arrangements in view are enumerated by the function

$$\frac{1}{(1)\,(2)\ldots(p_1)\,.\,(1)\,(2)\ldots(p_2)\ldots\ldots(1)\,(2)\ldots(p_m)}.$$

Hence

$$\frac{PF\,(p_1 p_2 \ldots p_m;\ \infty\,)}{(1)\,(2)\,(3)\ldots(\Sigma p)} = \frac{1}{(1)\,(2)\ldots(p_1)\,.\,(1)\,(2)\ldots(p_2)\ldots\ldots(1)\,(2)\ldots(p_m)},$$

or $$PF\,(p_1 p_2 \ldots p_m;\ \infty\,) = \frac{(1)\,(2)\,(3)\ldots(\Sigma p)}{(1)\,(2)\ldots(p_1)\,.\,(1)\,(2)\ldots(p_2)\ldots\ldots(1)\,(2)\ldots(p_m)},$$

the result anticipated in Vol. I, Sect. III, Ch. VI.

The expression should be compared with the number which enumerates the permutations of the assemblage

$$\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots\alpha_m^{p_m}.$$

The former becomes equal to the latter in the limit when x approaches unity.

454. When the part magnitude in the partitions is limited by the number n the partitions as defined in the last Article are enumerated by the expression

$$\frac{(n+1)\ldots(n+p_1)\,.\,(n+1)\ldots(n+p_2)\ldots\ldots(n+1)\ldots(n+p_m)}{(1)\ldots(p_1)\,.\,(1)\ldots(p_2)\ldots\ldots(1)\ldots(p_m)};$$

but employing the same reasoning as we did in the case of the lattice permutations, if $PF_s(p_1p_2\ldots p_m;\infty)$ be the sub-permutation function derived from the permutations which possess s major contacts, the enumerating function is *also*

$$\frac{(n+1)\ldots(n+\Sigma p)PF_0+(n)\ldots(n+\Sigma p-1)PF_1+\ldots+(n-\nu+1)\ldots(n-\nu+\Sigma p)PF_\nu}{(1)(2)\ldots(\Sigma p)}.$$

Hence

$$\begin{aligned}(n+1)\ldots(n+\Sigma p)PF_0&+(n)\ldots(n+\Sigma p-1)PF_1+\ldots+(n-\nu+1)\ldots(n-\nu+\Sigma p)PF_\nu\\ &=\{(n+1)\ldots(n+p_1)\,.\,(n+1)\ldots(n+p_2)\ldots\ldots(n+1)\ldots(n+p_m)\}\\ &\times\frac{(1)(2)\ldots(\Sigma p)}{(1)\ldots(p_1)\,.\,(1)\ldots(p_2)\ldots\ldots(1)\ldots(p_m)}\\ &=\{(n+1)\ldots(n+p_1)\,.\,(n+1)\ldots(n+p_2)\ldots\ldots(n+1)\ldots(n+p_m)\}\\ &\times PF(p_1p_2\ldots p_m;\infty);\end{aligned}$$

ν is equal to the maximum number of major contacts that can occur in a permutation of the assemblage. This was found in Vol. I to be $\Sigma p-p_1$.

When the lattice is complete and has as usual l columns and m rows

$$\nu=l(m-1).$$

455. Equating the two expressions for the generating function and giving n the values 0, 1, 2, 3, ... in succession, we find

$$PF_0=1,$$

$$\frac{(\Sigma p+1)}{(1)}+PF_1=\frac{(p_1+1)(p_2+1)\ldots(p_m+1)}{(1)^m},$$

$$\frac{(\Sigma p+1)(\Sigma p+2)}{(1)(2)}+\frac{(\Sigma p+1)}{(1)}PF_1+PF_2$$

$$=\frac{(p_1+1)(p_2+1)\ldots(p_m+1)}{(1)^m}\cdot\frac{(p_1+2)(p_2+2)\ldots(p_m+2)}{(2)^m},$$

etc.,

from which the general expression for PF_s is readily obtainable.

In fact

$$PF_0 = 1,$$

$$PF_1 = \frac{(p_1+1)(p_2+1)\dots(p_m+1)}{(1)^m} - \frac{(\Sigma p+1)}{(1)},$$

$$PF_2 = \frac{(p_1+1)(p_2+1)\dots(p_m+1)}{(1)^m} \cdot \frac{(p_1+2)(p_2+2)\dots(p_m+2)}{(2)^m}$$

$$- \frac{(\Sigma p+1)}{(1)} \cdot \frac{(p_1+1)(p_2+1)\dots(p_m+1)}{(1)^m} + x\frac{(\Sigma p)(\Sigma p+1)}{(1)(2)},$$

and in the expression of PF_s the last term is

$$(-)^s x^{\frac{1}{2}(s-1)s} \frac{(\Sigma p - s + 2)(\Sigma p - s + 3)\dots(\Sigma p+1)}{(1)(2)\dots(s)}.$$

456. If now from these formulæ we write down

$$PF_0 + PF_1 + \dots + PF_\nu \quad \text{where} \quad \nu = \Sigma p - p_1$$

we obtain a remarkable expression of $PF(p_1p_2\dots p_m; \infty)$ which we know otherwise has also the expression

$$\frac{(1)(2)\dots(\Sigma p)}{(1)(2)\dots(p_1)\,.\,(1)(2)\dots(p_2)\dots\dots(1)(2)\dots(p_m)}.$$

457. Putting, for a complete lattice, $p_1 = p_2 = \dots = p_m = l$, we find

$$PF(l; m; \infty) = \frac{(1)(2)\dots(lm)}{\{(1)(2)\dots(l)\}^m};$$

$$(n+1)\dots(n+lm)\,PF_0 + (n)(n+1)\dots(n+lm-1)\,PF_1 + \dots$$
$$+ (n+1-lm+l)\dots(n+l)\,PF_{lm-l}$$
$$= \{(n+1)(n+2)\dots(n+l)\}^m \cdot \frac{(1)(2)\dots(lm)}{\{(1)(2)\dots(l)\}^m}$$
$$= \left\{\frac{(n+1)(n+2)\dots(n+l)}{(1)(2)\dots(l)}\right\}^m (1)(2)\dots(lm);$$

and thence

$$PF_0 = 1,$$

$$PF_1 = \left\{\frac{(l+1)}{(1)}\right\}^m - \frac{(lm+1)}{(1)},$$

$$PF_2 = \left\{\frac{(l+1)(l+2)}{(1)(2)}\right\}^m - \frac{(lm+1)}{(1)}\left\{\frac{(l+1)}{(1)}\right\}^m + x\frac{(lm)(lm+1)}{(1)(2)},$$

$$PF_3 = \left\{\frac{(l+1)(l+2)(l+3)}{(1)(2)(3)}\right\}^m - \frac{(lm+1)}{(1)}\left\{\frac{(l+1)(l+2)}{(1)(2)}\right\}^m$$
$$+ x\frac{(lm)(lm+1)}{(1)(2)}\left\{\frac{(l+1)}{(1)}\right\}^m - x^3\frac{(lm-1)(lm)(lm+1)}{(1)(2)(3)},$$

etc.

458. As an example put $l = m = 2$, so that

$$PF(2; 2; \infty) = \frac{(1)(2)(3)(4)}{\{(1)(2)\}^2} = \frac{(3)(4)}{(1)(2)};$$

ν is here 2 and

$$PF_1 = \left\{\frac{(3)}{(1)}\right\}^2 - \frac{(5)}{(1)},$$

$$PF_2 = \left\{\frac{(3)(4)}{(1)(2)}\right\}^2 - \frac{(5)}{(1)}\left\{\frac{(3)}{(1)}\right\}^2 + x\frac{(4)(5)}{(1)(2)},$$

leading to the identity

$$\frac{(3)(4)}{(1)(2)} = \left[1 + \left\{\frac{(3)}{(1)}\right\}^2 + \left\{\frac{(3)(4)}{(1)(2)}\right\}^2\right] - \frac{(5)}{(1)}\left[1 + \left\{\frac{(3)}{(1)}\right\}^2\right] + x\frac{(4)(5)}{(1)(2)}.$$

459. Again putting $l = m = 3$, so that

$$PF(3; 3; \infty) = \frac{(1)(2)\ldots(9)}{\{(1)(2)(3)\}^3},$$

we find the identity

$$\begin{aligned}
&\frac{(1)(2)\ldots(9)}{\{(1)(2)(3)\}^3} \\
&= 1 + \left\{\frac{(4)}{(1)}\right\}^3 + \left\{\frac{(4)(5)}{(1)(2)}\right\}^3 + \left\{\frac{(4)(5)(6)}{(1)(2)(3)}\right\}^3 + \left\{\frac{(5)(6)(7)}{(1)(2)(3)}\right\}^3 \\
&+ \left\{\frac{(6)(7)(8)}{(1)(2)(3)}\right\}^3 + \left\{\frac{(7)(8)(9)}{(1)(2)(3)}\right\}^3 \\
&- \frac{(10)}{(1)}\left[1 + \left\{\frac{(4)}{(1)}\right\}^3 + \ldots + \left\{\frac{(6)(7)(8)}{(1)(2)(3)}\right\}^3\right] \\
&+ x\frac{(9)(10)}{(1)(2)}\left[1 + \left\{\frac{(4)}{(1)}\right\}^3 + \ldots + \left\{\frac{(5)(6)(7)}{(1)(2)(3)}\right\}^3\right] \\
&- x^3\frac{(8)(9)(10)}{(1)(2)(3)}\left[1 + \left\{\frac{(4)}{(1)}\right\}^3 + \ldots + \left\{\frac{(4)(5)(6)}{(1)(2)(3)}\right\}^3\right] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ x^{15}\frac{(5)(6)(7)(8)(9)(10)}{(1)(2)(3)(4)(5)(6)},
\end{aligned}$$

indicating clearly that the identities that arise from this theory are noticeable.

460. There is a remarkable simplification when $m = 2$, for then it can be shown that

$$PF_s(l; 2; \infty) = x^{s^2}\left\{\frac{(l)(l-1)\ldots(l-s+1)}{(1)(2)\ldots(s)}\right\}^2,$$

leading to the identity

$$\frac{(1)(2)\ldots(2l)}{\{(1)(2)\ldots(l)\}^2} = \sum_{s=0}^{s=l} x^{s^2}\left\{\frac{(l)(l-1)\ldots(l-s+1)}{(1)(2)\ldots(s)}\right\}^2,$$

which may be also written

$$GF(l;1;l) = \sum_{s=0}^{s=l} x^{s^2}\{GF(l-s;1;s)\}^2.$$

In fact, more generally, it will be found that

$$PF_s(p_1p_2;\infty) = x^{s^2}\frac{(\mathrm{p}_1)(\mathrm{p}_1-1)\dots(\mathrm{p}_1-\mathrm{s}+1)\,.\,(\mathrm{p}_2)(\mathrm{p}_2-1)\dots(\mathrm{p}_2-\mathrm{s}+1)}{\{(1)(2)\dots(\mathrm{s})\}^2}.$$

It has been shewn elsewhere (Vol. I, Sect. IV, Art. 146) that the number of permutations of the assemblage $\alpha^{p_1}\beta^{p_2}$ which possess s major contacts is given by the coefficient of $\lambda^s\alpha^{p_1}\beta^{p_2}$ in the product

$$(\alpha+\lambda\beta)^{p_1}(\alpha+\beta)^{p_2}.$$

From this product we can derive a function of x, viz.

$$(\alpha+\lambda\beta x)(\alpha+\lambda\beta x^2)\dots(\alpha+\lambda\beta x^{p_1})\,.\,(\beta+\alpha)(\beta+\alpha x)\dots(\beta+\alpha x^{p_2-1}),$$

and herein the coefficient of $\lambda^s\alpha^{p_1}\beta^{p_2}$ is

$$x^{s^2}\frac{(\mathrm{p}_1)(\mathrm{p}_1-1)\dots(\mathrm{p}_1-\mathrm{s}+1)\,.\,(\mathrm{p}_2)(\mathrm{p}_2-1)\dots(\mathrm{p}_2-\mathrm{s}+1)}{\{(1)(2)\dots(\mathrm{s})\}^2}.$$

For the first p_1 factors of the function of x may be written

$$\alpha^{p_1}+\lambda\alpha^{p_1-1}\beta x\frac{(\mathrm{p}_1)}{(1)}+\lambda^2\alpha^{p_1-2}\beta^2x^3\frac{(\mathrm{p}_1)(\mathrm{p}_1-1)}{(1)(2)}+\dots$$

$$+\lambda^s\alpha^{p_1-s}\beta^s x^{\binom{s+1}{2}}\frac{(\mathrm{p}_1)(\mathrm{p}_1-1)\dots(\mathrm{p}_1-\mathrm{s}+1)}{(1)(2)\dots(\mathrm{s})}+\dots,$$

and the last p_2 factors

$$\beta^{p_2}+\alpha\beta^{p_2-1}\frac{(\mathrm{p}_2)}{(1)}+\alpha^2\beta^{p_2-2}x\frac{(\mathrm{p}_2)(\mathrm{p}_2-1)}{(1)(2)}+\dots$$

$$+\alpha^s\beta^{p_2-s}x^{\binom{s}{2}}\frac{(\mathrm{p}_2)(\mathrm{p}_2-1)\dots(\mathrm{p}_2-\mathrm{s}+1)}{(1)(2)\dots(\mathrm{s})}+\dots,$$

so that the coefficient of $\alpha^{p_1}\beta^{p_2}$ is seen to be

$$1+\lambda x\frac{(\mathrm{p}_1)\,.\,(\mathrm{p}_2)}{\{(1)\}^2}+\lambda^2x^4\frac{(\mathrm{p}_1)(\mathrm{p}_1-1)\,.\,(\mathrm{p}_2)(\mathrm{p}_2-1)}{\{(1)(2)\}^2}+\dots$$

$$+\lambda^s x^{s^2}\frac{(\mathrm{p}_1)\dots(\mathrm{p}_1-\mathrm{s}+1)\,.\,(\mathrm{p}_2)\dots(\mathrm{p}_2-\mathrm{s}+1)}{\{(1)(2)\dots(\mathrm{s})\}^2}+\dots.$$

461. For the complete lattice the functions PF_s possess further elegant properties. Thus starting with the relation

$$\left\{\frac{(\mathrm{n}+1)(\mathrm{n}+2)\dots(\mathrm{n}+\mathrm{l})}{(1)(2)\dots(\mathrm{l})}\right\}^m$$

$$=\frac{(\mathrm{n}+1)\dots(\mathrm{n}+\mathrm{lm})PF_0+(\mathrm{n})\dots(\mathrm{n}+\mathrm{lm}-1)PF_1+\dots+(\mathrm{n}+1-\mathrm{lm}+\mathrm{l})\dots(\mathrm{n}+\mathrm{l})PF_{lm-l}}{(1)(2)\dots(\mathrm{lm})},$$

if in the left-hand side we substitute $-n-l-1$ for n we find that the effect is merely to multiply it by

$$x^{-lmn-\frac{1}{2}lm(l+1)},$$

whilst on the right-hand side the coefficient of PF_s is multiplied by

$$x^{-lmn+\frac{1}{2}lm(lm-2l-2s-1)}.$$

An identity thence arises and, putting therein $n=0, 1, 2, \ldots$ in succession, we find the relations

$$\begin{aligned} PF_{lm-l} &= x^{\frac{1}{2}m(m-1)l^2}\,PF_0, \\ PF_{lm-l-1} &= x^{\frac{1}{2}m(m-1)l^2-lm}\,PF_1, \\ &\vdots \\ PF_{lm-l-s} &= x^{\frac{1}{2}m(m-1)l^2-lms}\,PF_s; \end{aligned}$$

giving noteworthy algebraic identities.

462. We can find an expression for the sum

$$\sum_0^\infty g^n \frac{(n+1)\ldots(n+p_1)\,.\,(n+1)\ldots(n+p_2)\,\ldots\ldots\,(n+1)\ldots(n+p_m)}{(1)(2)\ldots(p_1)\,.\,(1)(2)\ldots(p_2)\,\ldots\ldots\,(1)(2)\ldots(p_m)},$$

for this is

$$\sum_0^\infty g^n \frac{(n+1)\ldots(n+\Sigma p)PF_0+(n)\ldots(n+\Sigma p-1)PF_1+\ldots+(n-\nu+1)\ldots(n-\nu+\Sigma p)PF_\nu}{(1)(2)\ldots(\Sigma p)}.$$

Now for

$$\sum_0^\infty g^n \frac{(n-s+1)(n-s+2)\ldots(n-s+\Sigma p)}{(1)(2)\ldots(\Sigma p)}PF_s$$

we have only to sum from $n=s$ to $n=\infty$ and the result is

$$\frac{g^s PF_s}{(1-g)(1-gx)(1-gx^2)\ldots(1-gx^{\Sigma p})}.$$

Hence the sum required is

$$\frac{1+gPF_1+g^2PF_2+\ldots+g^\nu PF_\nu}{(1-g)(1-gx)(1-gx^2)\ldots(1-gx^{\Sigma p})}.$$

463. The foregoing investigation establishes the identities

$$x\frac{(i)(j)}{(1)^2} = \frac{(i+1)(j+1)}{(1)^2} - \frac{(i+j+1)}{(1)};$$

$$x^4\frac{(i-1)(i)\,.\,(j-1)(j)}{(1)^2(2)^2}$$

$$= \frac{(i+1)(i+2)\,.\,(j+1)(j+2)}{(1)^2(2)^2} - \frac{(i+j+1)}{(1)}\frac{(i+1)(j+1)}{(1)^2}$$

$$+x\frac{(i+j)(i+j+1)}{(1)(2)};$$

and
$$x^9 \frac{(i-2)(i-1)(i)\,.\,(j-2)(j-1)(j)}{(1)^2(2)^2(3)^2}$$
$$= \frac{(i+1)(i+2)(i+3)\,.\,(j+1)(j+2)(j+3)}{(1)^2(2)^2(3)^2}$$
$$- \frac{(i+j+1)}{(1)} \frac{(i+1)(i+2)\,.\,(j+1)(j+2)}{(1)^2(2)^2}$$
$$+ x \frac{(i+j)(i+j+1)}{(1)(2)} \frac{(i+1)\,.\,(j+1)}{(1)^2}$$
$$- x^3 \frac{(i+j-1)(i+j)(i+j+1)}{(1)(2)(3)},$$
etc.;

and thence denoting
$$\frac{(i+1)(i+2)\ldots(i+s)\,.\,(j+1)(j+2)\ldots(j+s)}{(1)^2(2)^2\ldots(s)^2} \text{ by } A_{2s},$$
and
$$x^{\frac{1}{2}(s-1)s} \frac{(i+j-s+2)(i+j-s+3)\ldots(i+j+1)}{(1)(2)\ldots(s)} \text{ by } B_s,$$
we obtain
$$\frac{(j+1)(j+2)\ldots(j+i)}{(1)(2)\ldots(i)}$$
$$\begin{aligned} &= 1 + A_2 + A_4 + A_6 + \ldots + A_{2\nu} \\ &- B_1(1 + A_2 + A_4 + \ldots + A_{2\nu-2}) \\ &+ B_2(1 + A_2 + A_4 + \ldots + A_{2\nu-4}) \\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\vdots \\ &(-)^\nu B_\nu, \end{aligned}$$
where ν is the smallest of the numbers i, j.

SECTION X

COMPLETION OF THE THEORY OF SECTION IX

CHAPTER I

PLANE PARTITIONS WITH UNRESTRICTED PART MAGNITUDE

464. In this Section we determine the actual algebraic form of

$$L(p_1 p_2 \dots p_m ; n).$$

We first consider the particular case

$$L(p_1 p_2 \dots p_m ; \infty),$$

the lattice derived from the most general incomplete lattice. These determinations will of course lead to the expressions of

$$GF(p_1 p_2 \dots p_m ; n) \text{ and } GF(p_1 p_2 \dots p_m ; \infty).$$

An incomplete lattice is the graph either of the partition of a unipartite number or of a multipartite number. The partitions we enumerate are such that the parts, limited in magnitude to n, are placed at the nodes in such wise that there is a descending order of magnitude in each row and in each column. It is the most general case of plane partition as defined in this book.

As leading up to the method employed we will commence by determining the lattice function derived from two unequal rows, say

$$L(p_1 p_2 ; \infty).$$

The first step is to establish the relation

$$\begin{aligned} &L(p_1 p_2 ; \infty) \\ =\ &L(p_1, p_2 - 1 ; \infty) + x^{p_1+p_2-1} L(p_1 - 1, p_2 - 1 ; \infty) + x^{p_1+p_2-2} L(p_1 - 2, p_2 - 1 ; \infty) \\ &+ \dots + x^{2p_2} L(p_2, p_2 - 1 ; \infty). \end{aligned}$$

Consider the lattice permutations of the assemblage

$$\alpha^{p_1}\beta^{p_2} \text{ where } p_1 \geqslant p_2.$$

By reason of the fundamental property of a lattice permutation the letter α cannot occur more than $p_1 - p_2$ times repeated at the end of the permutation. Every permutation must terminate in one of the following ways:

$$\beta;\ \beta|\alpha;\ \beta|\alpha^2;\ \ldots\ \beta|\alpha^{p_1-p_2},$$

the dividing line as usual marking a major contact in the permutation. From these terminations components of the greater index arise, viz.:

$$0;\ p_1 + p_2 - 1;\ p_1 + p_2 - 2;\ \ldots\ 2p_2$$

respectively.

These give rise to factors in the power of x associated with a permutation, viz.:

$$1;\ x^{p_1+p_2-1};\ x^{p_1+p_2-2};\ \ldots\ x^{2p_2}$$

respectively.

In particular when the terminating letters are

$$\beta|\alpha^s,$$

the factor $x^{p_1+p_2-s}$ arises; the remaining letters are

$$\alpha^{p_1-s}\beta^{p_2-1},$$

and they may appear in any lattice permutation and produce a lattice function

$$L(p_1 - s, p_2 - 1; \infty).$$

Hence all permutations which terminate with $\beta\alpha^s$ $(s > 0)$ contribute

$$x^{p_1+p_2-s} L(p_1 - s, p_2 - 1; \infty)$$

to the lattice function $\qquad L(p_1, p_2; \infty)$.

When $s = 0$ the contribution is simply $L(p_1, p_2 - 1; \infty)$ since the power of x which is the factor is x^0 or 1.

We therefore have the relation

$$L(p_1, p_2; \infty) = L(p_1, p_2 - 1; \infty) + \sum_1^{p_1-p_2} x^{p_1+p_2-s} L(p_1 - s, p_2 - 1; \infty).$$

465. Now assume that

$$L(p_1, p_2 - 1; \infty) = \frac{(1)(2)\ldots(p_1 + p_2 - 1)}{(2)(3)\ldots(p_1 + 1).(1)(2)\ldots(p_2 - 1)} \cdot \frac{(p_1 + 1) - x(p_2 - 1)}{(1)}$$

for all values of p_1.

Then the functional equation leads to the relation

$$L(p_1 p_2; \infty)$$

$$= \frac{(1)(2)\dots(p_1+p_2-1)}{(2)(3)\dots(p_1+1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_1+1)-x(p_2-1)}{(1)}$$

$$+ x^{p_1+p_2-1} \frac{(1)(2)\dots(p_1+p_2-2)}{(2)(3)\dots(p_1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_1)-x(p_2-1)}{(1)}$$

$$+ x^{p_1+p_2-2} \frac{(1)(2)\dots(p_1+p_2-2)}{(2)(3)\dots(p_1-1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_1-1)-x(p_2-1)}{(1)}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ x^{2p_2} \frac{(1)(2)\dots(2p_2-1)}{(2)(3)\dots(p_2+1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_2+1)-x(p_2-1)}{(1)}.$$

The right-hand side has $p_1 - p_2 + 1$ terms; assume that the sum of the last k terms may be written

$$x^{2p_2} \frac{(1)(2)\dots(2p_2+k-1)}{(1)(2)\dots(p_2+k).(1)(2)\dots(p_2)} (k),$$

an assumption which it is easy to see is justified when $k = 1$.

The $(k+1)$th term from the end is the $(p_1 - p_2 - k + 1)$th from the beginning and this is

$$x^{2p_2+k} \frac{(1)(2)\dots(2p_2+k-1)}{(1)(2)\dots(p_2+k-1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_2+k+1)-x(p_2-1)}{(1)}.$$

Adding this to the sum of the last k terms we find on simplification

$$x^{2p_2} \frac{(1)(2)\dots(2p_2+k)}{(1)(2)\dots(p_2+k+1).(1)(2)\dots(p_2)} (k+1),$$

a result which justifies the assumption.

Hence putting $k = p_1 - p_2$ we find

$$L(p_1 p_2; \infty)$$

$$= \frac{(1)(2)\dots(p_1+p_2-1)}{(2)(3)\dots(p_1+1).(1)(2)\dots(p_2-1)} \cdot \frac{(p_1+1)-x(p_2-1)}{(1)}$$

$$+ x^{2p_2} \frac{(1)(2)\dots(p_1+p_2-1)}{(1)(2)\dots(p_1).(1)(2)\dots(p_2)} \cdot (p_1-p_2)$$

$$= \frac{(1)(2)\dots(p_1+p_2)}{(2)(3)\dots(p_1+1).(1)(2)\dots(p_2)} \cdot \frac{(p_1+1)-x(p_2)}{(1)}.$$

This result being true when $p_2 = 0$ is established universally.

466. It must reduce when x is put equal to unity to the number which enumerates the lattice permutations of the assemblage

$$\alpha^{p_1}\beta^{p_2}.$$

This number is

$$\frac{(p_1+p_2)!}{(p_1+1)!\,p_2!}(p_1-p_2+1),$$

and the resemblance between the algebraic function and the enumerating number at once strikes an observer.

467. We therefore seek in the general case to always have this resemblance in evidence. This idea leads to a very interesting arrangement of the work.

The number which enumerates the lattice permutations of the assemblage

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_m^{p_m}$$

may from Sect. III, Ch. v be written

$$\frac{(1)(2)\ldots(\Sigma p)}{(m)\ldots(p_1+m-1).(m-1)\ldots(p_2+m-2)\ldots\ldots(1)\ldots(p_m)}$$

$$\times\frac{\prod\limits_{s,t}(p_s-p_t+t-s)}{\prod\limits_{s,t}(t-s)},$$

wherein $t>s$ and the continued product has reference to every pair of numbers that can be selected from the first m natural numbers.

We now say that we shall express the lattice function

$$L(p_1p_2\ldots p_m;\infty)$$

in the form

$$\frac{\mathbf{(1)\,(2)\ldots(\Sigma p)}}{\mathbf{(m)\ldots(p_1+m-1).(m-1)\ldots(p_2+m-2)\ldots\ldots(1)\ldots(p_m)}}$$

$$\times IL(p_1p_2\ldots p_m;\infty),$$

where the algebraic fraction first written is a fixed factor of the lattice function and the remaining factor has to be determined.

The fixed factor before us is called the Outer Lattice Function and written

$$OL(p_1p_2\ldots p_m;\infty).$$

The remaining factor is called the Inner Lattice Function and is written

$$IL(p_1p_2\ldots p_m;\infty).$$

The expression of the outer lattice function is in every case known. That of the inner function is unknown and has to be determined.

We have

$$L(p_1p_2\ldots p_m;\infty)=OL(p_1p_2\ldots p_m;\infty)\times IL(p_1p_2\ldots p_m;\infty).$$

Above we have arrived at our first result

$$IL(p_1p_2;\infty)=\frac{\mathbf{(p_1+1)}-x\,\mathbf{(p_2)}}{\mathbf{(1)}}.$$

468. A principal object of this Section is the determination of the algebraic expression of the inner lattice function for the general incomplete lattice. A particular object is to prove that in the case of the complete lattice the inner lattice function is unity, viz.:

$$IL(l; m; \infty) = 1.$$

In that case the outer lattice function is identical with the lattice function.

It has been already proved that if we substitute for a lattice the conjugate lattice, the lattice function as well as all of the sub-lattice functions are unchanged. This is only true of the outer and inner lattice functions in special cases. As a general rule they both change when such a change is made in the lattice. The product of the outer and inner lattice functions, being equal to the lattice function, necessarily remains unchanged under the circumstances stated.

We have as a consequence of the new arrangements

$$GF(p_1 p_2 \ldots p_m; \infty) = \frac{OL(p_1 p_2 \ldots p_m; \infty)}{(1)(2)\ldots(\Sigma p)} . IL(p_1 p_2 \ldots p_m; \infty),$$

where

$$OL(p_1 p_2 \ldots p_m; \infty)$$

$$= \frac{(1)(2)\ldots(\Sigma p)}{(m)\ldots(p_1+m-1).(m-1)\ldots(p_2+m-2)\ldots\ldots(1)\ldots(p_m)},$$

so that

$$GF(p_1 p_2 \ldots p_m; \infty)$$

$$= \frac{IL(p_1 p_2 \ldots p_m; \infty)}{(m)\ldots(p_1+m-1).(m-1)\ldots(p_2+m-2)\ldots\ldots(1)\ldots(p_m)}.$$

The only general fact we at present know about the inner lattice function is that the sum of its coefficients is equal to

$$\prod_{s,t} \frac{p_s - p_t + t - s}{t - s}.$$

When this question has been completely answered, the next step will be to consider the problem when the part magnitude is restricted.

469. So far we have established the formula

$$GF(p_1 p_2; \infty) = \frac{\dfrac{(p_1+1) - x(p_2)}{(1)}}{(2)(3)\ldots(p_1+1).(1)(2)\ldots(p_2)},$$

and its particular case

$$GF(pp; \infty) = GF(p; 2; \infty) = \frac{1}{(1)\{(2)(3)\ldots(p)\}^2(p+1)},$$

by considering the derivation of the lattice function from lattice permutations. We found a functional equation connecting lattice functions and we might have then proceeded to form a functional equation connecting the corresponding inner lattice functions. We might adopt the same procedure

in the further study of the subject, but there is an alternative. We may proceed from the solid graph of a plane partition to form a functional equation between the generating functions which enumerate plane partitions and thence deduce equations connecting the lattice and inner lattice functions. This is the more convenient course. Suppose that the lattice, at the nodes of which the parts of the partition are placed, has three unequal rows comprising p_1, p_2 and p_3 nodes respectively, where of course $p_1 \geqslant p_2 \geqslant p_3$.

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For a moment we suppose the part magnitude to be restricted by the number n.

Subject to the parts being in descending order of magnitude in each row and in each column, in every plane partition each node is either occupied by zero (that is, is unoccupied) or by some number greater than zero and not greater than n.

In certain partitions *every* node is occupied; such partitions may be constructed by

(i) placing a unit at each node,

(ii) superposing every partition enumerated by $GF(p_1p_2p_3;\, n-1)$.

Such partitions will be termed "full-based" partitions.

They are clearly enumerated by

$$x^{p_1+p_2+p_3}\, GF(p_1p_2p_3;\, n-1),$$

the factor $x^{p_1+p_2+p_3}$ arising from the presence of the base which has every node occupied by a unit.

Every lattice that can be formed from a given lattice by obliterating nodes may be said to be "contained" in the given lattice. Among such contained lattices is reckoned the given lattice itself.

Now suppose that a lattice having rows containing p_1', p_2', p_3' nodes is contained in the given lattice; it is obvious that some of the plane partitions we seek to enumerate will be full-based upon such contained lattice and will be enumerated by

$$x^{p_1'+p_2'+p_3'}\, GF(p_1'p_2'p_3';\, n-1).$$

Speaking of lattices contained in a given lattice is equivalent, since a lattice is the graph of a line partition, to speaking of line partitions contained in a given line partition.

We thus find that

$$GF(p_1p_2p_3;\, n) = \Sigma x^{p_1'+p_2'+p_3'}\, GF(p_1'p_2'p_3';\, n-1),$$

the sign of summation indicating every lattice $p_1'p_2'p_3'$ contained in the lattice $p_1p_2p_3$.

470. This relation shews the derivation of the generating function when the highest part is n from generating functions when the highest part is $n-1$. When there is no restriction upon the part magnitude

$$GF(p_1p_2p_3;\ \infty) = \Sigma x^{p_1'+p_2'+p_3'}\, GF(p_1'p_2'p_3';\ \infty),$$

and generally

$$GF(p_1p_2 \dots p_m;\ \infty) = \Sigma x^{p_1'+p_2'+\dots+p'_m}\, GF(p_1'p_2' \dots p'_m;\ \infty),$$

leading to

$$\frac{L(p_1p_2 \dots p_m;\ \infty)}{(1)\,(2) \dots (\Sigma p)} = \Sigma x^{\Sigma p'} \frac{L(p_1'p_2' \dots p'_m;\ \infty)}{(1)\,(2) \dots (\Sigma p')};$$

$$\frac{IL(p_1p_2 \dots p_m;\ \infty)}{(m) \dots (p_1+m-1)\,.\,(m-1) \dots (p_2+m-2) \dots\dots (1) \dots (p_m)}$$

$$= \Sigma x^{\Sigma p'} \frac{IL(p_1'p_2' \dots p'_m;\ \infty)}{(m) \dots (p_1'+m-1)\,.\,(m-1) \dots (p_2'+m-2) \dots\dots (1) \dots (p'_m)}.$$

It must not be overlooked that in these relations the lattice $p_1p_2 \dots p_m$ is included under the sign of summation on the right-hand side. Also in using the last formula that m may be $1, 2, \dots m-1$ or m, according to the number of rows in the contained lattice. Also that unity arising from the absence of nodes is included under the sign of summation as usual in all partition formulæ.

As an example we find that

$$(1-x^4)\frac{IL(22;\ \infty)}{(1)\,(2)^2\,(3)} = x^3 \frac{IL(21;\ \infty)}{(1)\,(2)\,(3)} + x^2 \frac{IL(11;\ \infty)}{(1)\,(2)} + x^2 \frac{IL(2;\ \infty)}{(1)\,(2)}$$
$$+ x \frac{IL(1;\ \infty)}{(1)} + 1,$$

and since

$$IL(21;\ \infty) = 1 + x^2, \quad IL(11;\ \infty) = IL(2;\ \infty) = IL(1;\ \infty) = 1,$$

we verify that

$$IL(22;\ \infty) = 1.$$

In this case the lattices contained in

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}$$

are

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array} \qquad \begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array} \qquad \begin{array}{c} \bullet \\ \bullet \end{array} \qquad \begin{array}{cc} \bullet & \bullet \end{array} \qquad \bullet$$

and the zero lattice counting as unity under the summation sign.

If from the plane partitions enumerated by

$$GF(p_1p_2p_3;\ n),$$

we subtract those enumerated by

$$x^{p_1+p_2+p_3}\, GF(p_1p_2p_3;\ n-1),$$

we have remaining, in the case of three unequal rows, partitions which *include* those enumerated by each of the three generating functions

$$GF(p_1-1\ p_2p_3;\ n),\quad GF(p_1\ p_2-1\ p_3;\ n),\quad GF(p_1p_2\ p_3-1;\ n),$$

and these, by a well-known principle of the combinatory analysis (see Whitworth's *Choice and Chance*, Fourth Edition, pp. 73 *et seq.*), are enumerated by

$$\begin{aligned}&GF(p_1-1\ p_2p_3;\ n)+GF(p_1p_2-1\ p_3;\ n)+GF(p_1p_2\ p_3-1;\ n)\\ &-GF(p_1-1\ p_2-1\ p_3;\ n)-GF(p_1-1\ p_2\ p_3-1;\ n)-GF(p_1\ p_2-1\ p_3-1;\ n)\\ &+GF(p_1-1\ p_2-1\ p_3-1;\ n).\end{aligned}$$

The reader will see without difficulty that in the functions of the first line of this expression the partitions enumerated by each of the functions in the second line are enumerated twice over. We therefore subtract these functions each once, but in doing so we find that the partitions enumerated by the function in the third line have been omitted altogether. We therefore add once the function in the third line.

Hence the functional equation

$$\begin{aligned}&GF(p_1p_2p_3;\ n)-x^{p_1+p_2+p_3}GF(p_1p_2p_3;\ n-1)\\ &=GF(p_1-1\ p_2p_3;\ n)+GF(p_1\ p_2-1\ p_3;\ n)+GF(p_1p_2\ p_3-1;\ n)\\ &-GF(p_1-1\ p_2-1\ p_3;\ n)-GF(p_1-1\ p_2\ p_3-1;\ n)-GF(p_1\ p_2-1\ p_3-1;\ n)\\ &+GF(p_1-1\ p_2-1\ p_3-1;\ n).\end{aligned}$$

471. In the general case of m unequal rows we have the functional equation which is elegantly expressed in the following manner:

Let θ_s be a symbol such that

$$\theta_s\,GF(p_1p_2\ldots p_s\ldots p_m;\ n)=GF(p_1p_2\ldots p_s-1\ldots p_m;\ n),$$

then

$$(1-\theta_1)(1-\theta_2)\ldots(1-\theta_m)\,GF(p_1p_2\ldots p_m;\ n)=x^{\Sigma p}\,GF(p_1p_2\ldots p_m;\ n-1).$$

472. The formula may be modified in the direction of simplification when the rows are not all different.

For suppose $p_3=p_2$

$$\begin{array}{lllllll}\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&\bullet&&\\ \bullet&\bullet&\bullet&\bullet&\bullet&&\end{array}$$

we must see how many nodes may be *singly* obliterated so as to leave a contained lattice. It is clear that in this case the right-hand nodes of the first and third rows may be detached but no other node. Reasoning as before we are led to the functional equation

$$\begin{aligned}&GF(p_1p_2p_2;\ n)-x^{p_1+2p_2}GF(p_1p_2p_2;\ n-1)\\ &=GF(p_1-1\ p_2p_2;\ n)+GF(p_1p_2\ p_2-1;\ n)\\ &-GF(p_1-1\ p_2\ p_2-1;\ n),\end{aligned}$$

which may be compared with the equation appertaining to a lattice of two unequal rows:

$$GF(p_1p_2;\, n) - x^{p_1+p_2}\, GF(p_1p_2;\, n-1)$$
$$= GF(p_1-1\, p_2;\, n) + GF(p_1\, p_2-1;\, n) - GF(p_1-1\, p_2-1;\, n).$$

Similarly we derive the equations

$$GF(p_1p_1p_2;\, n) - x^{2p_1+p_2}\, GF(p_1p_1p_2;\, n-1)$$
$$= GF(p_1\, p_1-1\, p_2;\, n) + GF(p_1p_1\, p_2-1;\, n) - GF(p_1\, p_1-1\, p_2-1;\, n);$$
$$GF(p_1p_1p_1;\, n) - x^{3p_1}\, GF(p_1p_1p_1;\, n-1) = GF(p_1p_1\, p_1-1;\, n);$$

and also

$$GF(p_1^m;\, n) - x^{mp_1}\, GF(p_1^m;\, n-1) = GF(p_1^{m-1}\, p_1-1;\, n).$$

473. When the part magnitude is unrestricted we have

$$(\mathbf{p_1}+\mathbf{p_2})\, GF(p_1p_2;\, \infty) = GF(p_1-1\, p_2;\, \infty) + GF(p_1\, p_2-1;\, \infty)$$
$$- GF(p_1-1\, p_2-1;\, \infty);$$

$$(\mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3})\, GF(p_1p_2p_3;\, \infty)$$
$$= GF(p_1-1\, p_2p_3;\, \infty) + GF(p_1\, p_2-1\, p_3;\, \infty) + GF(p_1p_2\, p_3-1;\, \infty)$$
$$- GF(p_1-1\, p_2-1\, p_3;\, \infty) - GF(p_1-1\, p_2\, p_3-1;\, \infty) - GF(p_1\, p_2-1\, p_3-1;\, \infty)$$
$$+ GF(p_1-1\, p_2-1\, p_3-1;\, \infty);$$

$$(\mathbf{\Sigma p})\, GF(p_1p_2\ldots p_m;\, \infty) = \{1-(1-\theta_1)(1-\theta_2)\ldots(1-\theta_m)\}\, GF(p_1p_2\ldots p_m;\, \infty).$$

In general when a lattice has some equal rows so that $p_s^{\pi_s}$ appears in the specification of the lattice the symbol θ_s acts in the manner

$$\theta_s\, GF(p_1\ldots p_s^{\pi_s}\ldots p_m;\, \infty) = GF(p_1\ldots p_s^{\pi_s-1}\, p_s-1\ldots p_m;\, \infty),$$

so that whatever equalities of rows may exist the symbolic expression is available.

474. The next step is the deduction of the corresponding relation between lattice functions.

Substitution from the relation

$$GF(p_1p_2\ldots p_m;\, \infty) = \frac{L(p_1p_2\ldots p_m;\, \infty)}{(\mathbf{1})\,(\mathbf{2})\ldots(\mathbf{\Sigma p})},$$

gives

$$L(pp;\, \infty) = L(p\, p-1;\, \infty);$$
$$L(p_1p_2;\, \infty) = L(p_1-1\, p_2;\, \infty) + L(p_1\, p_2-1;\, \infty)$$
$$- (\mathbf{p_1}+\mathbf{p_2}-\mathbf{1})\, L(p_1-1\, p_2-1;\, \infty);$$
$$L(ppp;\, \infty) = L(pp\, p-1;\, \infty);$$
$$L(p_1p_1p_2;\, \infty) = L(p_1\, p_1-1\, p_2;\, \infty) + L(p_1p_1\, p_2-1;\, \infty)$$
$$- (\mathbf{2p_1}+\mathbf{p_2}-\mathbf{1})\, L(p_1\, p_1-1\, p_2-1;\, \infty);$$

$$L(p_1p_2p_2;\ \infty) = L(p_1-1\ p_2p_2;\ \infty) + L(p_1p_2\ p_2-1;\ \infty)$$
$$- (\mathsf{p_1+2p_2-1})\ L(p_1-1\ p_2\ p_2-1;\ \infty);$$

$$L(p_1p_2p_3;\ \infty) = L(p_1-1\ p_2p_3;\ \infty) + L(p_1\ p_2-1\ p_3;\ \infty) + L(p_1p_2\ p_3-1;\ \infty)$$
$$- (\mathsf{p_1+p_2+p_3-1})\ \{L(p_1-1\ p_2-1\ p_3;\ \infty)$$
$$+ L(p_1-1\ p_2\ p_3-1;\ \infty) + L(p_1\ p_2-1\ p_3-1;\ \infty)\}$$
$$+ (\mathsf{p_1+p_2+p_3-1})\ (\mathsf{p_1+p_2+p_3-2})\ L(p_1-1\ p_2-1\ p_3-1;\ \infty).$$

In the case of m unequal rows if θ_s be the symbol before defined and X a new symbol such that

$$(\Sigma p)(\Sigma p-1)\ldots(\Sigma p-s+1) = X^s,$$

we have in general

$$(1-X)\,L(p_1p_2\ldots p_m;\ \infty)$$
$$= (1-\theta_1X)(1-\theta_2X)\ldots(1-\theta_mX)\,L(p_1p_2\ldots p_m;\ \infty).$$

475. We now proceed to the equation satisfied by the inner lattice functions. Substituting from the relation

$$L(p_1p_2\ldots p_m;\ \infty)$$
$$= \frac{(\mathsf{1})\ (\mathsf{2})\ \ldots\ (\mathsf{\Sigma p})\ .\ IL(p_1p_2\ldots p_m;\ \infty)}{(\mathsf{m})\ \ldots\ (\mathsf{p_1+m-1})\ .\ (\mathsf{m-1})\ \ldots\ (\mathsf{p_2+m-2})\ \ldots\ldots\ (\mathsf{1})\ \ldots\ (\mathsf{p_m})},$$

we find

$$(\mathsf{2p})\ IL(pp;\ \infty) = (\mathsf{p})\ IL(p\ p-1;\ \infty);$$

$$(\mathsf{p_1+p_2})\ IL(p_1p_2;\ \infty)$$
$$= (\mathsf{p_1+1})\ IL(p_1-1\ p_2;\ \infty) + (\mathsf{p_2})\ IL(p_1\ p_2-1;\ \infty)$$
$$- (\mathsf{p_1+1})\ (\mathsf{p_2})\ IL(p_1-1\ p_2-1;\ \infty);$$

$$(\mathsf{3p})\ IL(ppp;\ \infty) = (\mathsf{p})\ IL(pp\ p-1;\ \infty);$$

$$(\mathsf{2p_1+p_2})\ IL(p_1p_1p_2;\ \infty)$$
$$= (\mathsf{p_1+1})\ IL(p_1\ p_1-1\ p_2;\ \infty) + (\mathsf{p_2})\ IL(p_1p_1\ p_2-1;\ \infty)$$
$$- (\mathsf{p_1+1})\ (\mathsf{p_2})\ IL(p_1\ p_1-1\ p_2-1;\ \infty);$$

$$(\mathsf{p_1+2p_2})\ IL(p_1p_2p_2;\ \infty)$$
$$= (\mathsf{p_1+2})\ IL(p_1-1\ p_2p_2;\ \infty) + (\mathsf{p_2})\ IL(p_1p_2\ p_2-1;\ \infty)$$
$$- (\mathsf{p_1+2})\ (\mathsf{p_2})\ IL(p_1-1\ p_2\ p_2-1;\ \infty);$$

$$(\mathsf{p_1+p_2+p_3})\ IL(p_1p_2p_3;\ \infty)$$
$$= (\mathsf{p_1+2})\ IL(p_1-1\ p_2p_3;\ \infty) + (\mathsf{p_2+1})\ IL(p_1\ p_2-1\ p_3;\ \infty)$$
$$+ (\mathsf{p_3})\ IL(p_1p_2\ p_3-1;\ \infty) - (\mathsf{p_1+2})\ (\mathsf{p_2+1})\ IL(p_1-1\ p_2-1\ p_3;\ \infty)$$
$$- (\mathsf{p_1+2})\ (\mathsf{p_3})\ IL(p_1-1\ p_2p_3-1;\ \infty) - (\mathsf{p_2+1})\ (\mathsf{p_3})\ IL(p_1\ p_2-1\ p_3-1;\ \infty)$$
$$+ (\mathsf{p_1+2})\ (\mathsf{p_2+1})\ (\mathsf{p_3})\ IL(p_1-1\ p_2-1\ p_3-1;\ \infty).$$

476. In general, for m unequal rows, if ϕ_s be a symbol such that

$$\phi_s IL(p_1p_2 \dots p_m;\ \infty) = (\mathbf{p_s + m - s})\, IL(p_1p_2 \dots p_s - 1 \dots p_m;\ \infty),$$

$$x^{\Sigma p} IL(p_1p_2 \dots p_m;\ \infty) = (1-\phi_1)(1-\phi_2)\dots(1-\phi_m)\, IL(p_1p_2 \dots p_m;\ \infty).$$

When the specification of the lattice involves repetitions of letters the equations satisfied by the lattice and inner lattice functions are

$$(1-X)\, L(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty)$$

$$= (1-\theta_1X)(1-\theta_2X)\dots(1-\theta_mX)\, L(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty),$$

where

$$\theta_s L(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty) = L(p_1^{\pi_1}p_2^{\pi_2}\dots p_s^{\pi_s - 1}p_s - 1 \dots p_m^{\pi_m};\ \infty),$$

and

$$x^{\Sigma p} IL(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty)$$

$$= (1-\phi_1)(1-\phi_2)\dots(1-\phi_m)\, IL(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty),$$

where

$$\phi_s L(p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m};\ \infty)$$

$$= (\mathbf{p_s + \pi_{s+1} + \pi_{s+2} + \dots + \pi_m})\, IL(p_1^{\pi_1}p_2^{\pi_2}\dots p_s^{\pi_s - 1}p_s - 1 \dots p_m^{\pi_m};\ \infty).$$

The reader should verify the validity of the general formulæ in the case of the particular examples $m = 1$, $m = 2$ and $m = 3$ given at length above. He may be reminded also that throughout the investigation we employ the notation $1 - x^s = (\mathbf{s})$, the letters or figures in the bracket being in Clarendon type.

477. The following investigation into the form of the inner lattice function has reference in the first instance to lattices with unequal rows. Later the results will be shewn to be applicable to lattices in general. In the case of the lattice of two unequal rows we have established the theorem

$$IL(p_1p_2;\ \infty) = \frac{(\mathbf{p_1+1})}{(\mathbf{1})} - x\frac{(\mathbf{p_2})}{(\mathbf{1})};$$

the sum of the coefficients of this integral algebraic function is visibly $p_1 - p_2 + 1$ as it should be.

Now we know that the sum of the coefficients in $IL(p_1p_2p_3;\ \infty)$ is $(p_1 - p_2 + 1)(p_1 - p_3 + 2)(p_2 - p_3 + 1)$ and it might be conjectured that the form of $IL(p_1p_2p_3;\ \infty)$ is

$$\left\{\frac{(\mathbf{p_1+1})}{(\mathbf{1})} - x\frac{(\mathbf{p_2})}{(\mathbf{1})}\right\}\left\{\frac{(\mathbf{p_1+2})}{(\mathbf{2})} - x^2\frac{(\mathbf{p_3})}{(\mathbf{2})}\right\}\left\{\frac{(\mathbf{p_2+1})}{(\mathbf{1})} - x\frac{(\mathbf{p_3})}{(\mathbf{1})}\right\},$$

but this function is quickly seen neither to satisfy the functional equation nor to be verifiable in simple cases.

It must also be noticed that the lattice function $L(p_1p_2 \dots p_m;\ \infty)$ becomes equal to $L(p_1p_2 \dots p_{m-1};\ \infty)$ on putting $p_m = 0$, but that we have *no* reason to conclude that the inner lattice function $IL(p_1p_2 \dots p_m;\ \infty)$ becomes $IL(p_1p_2 \dots p_{m-1};\ \infty)$ on putting $p_m = 0$.

In fact it does not do so, as can be verified at once by comparing

$$\frac{(\mathbf{p}_1+1)}{(1)} - x\frac{(\mathbf{p}_2)}{(1)},$$

the expression of $IL(p_1p_2; \infty)$, with unity which is the expression of $IL(p_1; \infty)$.

Now we know also that $GF(p_1p_2 \ldots p_m; \infty)$ becomes $GF(p_1p_2 \ldots p_{m-1}; \infty)$ on putting $p_m = 0$ and we find that if we take the formula

$$GF(p_1p_2p_3; \infty) = \frac{IL(p_1p_2p_3; \infty)}{(3)(4)\ldots(\mathbf{p}_1+2).(2)(3)\ldots(\mathbf{p}_2+1).(1)(2)\ldots(\mathbf{p}_3)},$$

and substitute for $IL(p_1p_2p_3; \infty)$ the product of three factors above written, the putting of $p_3 = 0$ *does* produce the correct expression of $GF(p_1p_2; \infty)$. We gather from this fact that we are justified in concluding that we may put

$$IL(p_1p_2p_3; \infty)$$

$$= \left\{\frac{(\mathbf{p}_1+1)}{(1)} - x\frac{(\mathbf{p}_2)}{(1)}\right\}\left\{\frac{(\mathbf{p}_1+2)}{(2)} - x^2\frac{(\mathbf{p}_3)}{(2)}\right\}\left\{\frac{(\mathbf{p}_2+1)}{(1)} - x\frac{(\mathbf{p}_3)}{(1)}\right\} + (\mathbf{p}_3)\,F(p_1p_2p_3; \infty),$$

because $p_3 = 0$ causes $(\mathbf{p}_3)$ to vanish.

$F(p_1p_2p_3; \infty)$ is a function to be determined.

We further conclude, by working out a few particular cases, that a probable result is

$$F(p_1p_2p_3; \infty) = \frac{1}{(2)}\{x^{p_2+3}(\mathbf{p}_1-\mathbf{p}_2) - x^{p_3+2}(\mathbf{p}_2-\mathbf{p}_3)\},$$

and on trial it is found, the work being too tedious to reproduce here, that the expression

$$\left\{\frac{(\mathbf{p}_1+1)}{(1)} - x\frac{(\mathbf{p}_2)}{(1)}\right\}\left\{\frac{(\mathbf{p}_1+2)}{(2)} - x^2\frac{(\mathbf{p}_3)}{(2)}\right\}\left\{\frac{(\mathbf{p}_2+1)}{(1)} - x\frac{(\mathbf{p}_3)}{(1)}\right\}$$
$$+\frac{(\mathbf{p}_3)}{(2)}\{x^{p_2+3}(\mathbf{p}_1-\mathbf{p}_2) - x^{p_3+2}(\mathbf{p}_2-\mathbf{p}_3)\}$$

does, as a fact, satisfy the functional equation.

478. The actual forms of the inner lattice functions for the cases of two and three rows will prove of much advantage in the investigation to which we now proceed.

In the equation $\qquad (\mathbf{p}_1+\mathbf{p}_2)\,IL(p_1p_2; \infty)$

$$= (\mathbf{p}_1+1)\,IL(p_1-1\,p_2; \infty) + (\mathbf{p}_2)\,IL(p_1\,p_2-1; \infty)$$
$$-(\mathbf{p}_1+1)(\mathbf{p}_2)\,IL(p_1-1\,p_2-1; \infty)$$

put

$$x^{-p_1}\{IL(p_1p_2; \infty) - (\mathbf{p}_1+1)\,IL(p_1-1\,p_2; \infty)\} = V_1(p_1p_2; \infty),$$

then
$$(\mathbf{p_1+p_2})\, V_1(p_1p_2;\infty)$$
$$=(\mathbf{p_1+1})\, V_1(p_1-1\; p_2;\infty)+(\mathbf{p_2})\, V_1(p_1\; p_2-1;\infty)$$
$$-(\mathbf{p_1+1})(\mathbf{p_2})\, V_1(p_1-1\; p_2-1;\infty),$$
an equation of the same form as that satisfied by $IL\,(p_1p_2;\infty)$.

Hence we conclude that if $IL\,(p_1p_2;\infty)$ is a solution of tl

$$x^{-p_1}\{IL\,(p_1p_2;\infty)-(\mathbf{p_1+1})\, IL\,(p_1-1\,p_2;\infty)\}$$

is also a solution.

It is convenient to exhibit the new solution by means of an (performed upon the original solution.

We therefore write

$$O_{p_1}\, IL\,(p_1p_2;\infty)=V_1\,(p_1p_2;\infty).$$

479. Again put

$$x^{-p_2}\{IL\,(p_1p_2;\infty)-(\mathbf{p_2})\, IL\,(p_1\,p_2-1;\infty)\}=V_2(p_1p_2;\infty),$$

and we find that

$$(\mathbf{p_1+p_2})\, V_2(p_1p_2;\infty)$$
$$=(\mathbf{p_1+1})\, V_2(p_1-1\; p_2;\infty)+(\mathbf{p_2})\, V_2(p_1\; p_2-1;\infty)$$
$$-(\mathbf{p_1+1})(\mathbf{p_2})\, V_2(p_1-1\; p_2-1;\infty),$$

again an equation of the same form as that satisfied by $IL\,(p_1p_2;\infty)$.

Therefore another solution is $V_2\,(p_1p_2;\infty)$ which we may represent by

$$O_{p_2}\, IL\,(p_1p_2;\infty).$$

Further suppose that $O_{p_1p_2}$ is an operation such that

$$O_{p_1p_2}\, IL\,(p_1p_2;\infty)$$
$$=x^{-p_1-p_2}\{IL\,(p_1p_2;\infty)-(\mathbf{p_1+1})\, IL\,(p_1-1\; p_2;\infty)-(\mathbf{p_2})\, IL\,(p_1\; p_2-1;\infty)$$
$$+(\mathbf{p_1+1})(\mathbf{p_2})\, IL\,(p_1-1\; p_2-1;\infty)\},$$

so that from the functional equation itself

$$O_{p_1p_2}\, IL\,(p_1p_2;\infty)=IL\,(p_1p_2;\infty),$$

or
$$O_{p_1p_2}=1.$$

Moreover if we operate with O_{p_2} upon $V_1(p_1p_2;\infty)$ we reproduce $IL\,(p_1p_2;\infty)$, as may be verified in two lines; so that we have the relations

$$O_{p_1}O_{p_2}=O_{p_1p_2}=1.$$

We now apply these operators to the known solution of the functional equation which, discarding the non-essential factor $\frac{1}{(\mathbf{1})}$, we may take to be

$$(\mathbf{p_1+1})-x\,(\mathbf{p_2}).$$

We find
$$O_{p_1}\{(\mathbf{p}_1+\mathbf{1})-x\,(\mathbf{p}_2)\}=(\mathbf{p}_1+\mathbf{1})-x^2\,(\mathbf{p}_2),$$
giving us another solution $(\mathbf{p}_1+\mathbf{1})-x^2\,(\mathbf{p}_2)$; we can now operate again, and operating $s-1$ times successively we obtain
$$O_{p_1}^{s-1}\{(\mathbf{p}_1+\mathbf{1})-x\,(\mathbf{p}_2)\}=(\mathbf{p}_1+\mathbf{1})-x^s\,(\mathbf{p}_2).$$

Similarly operating with O_{p_2} we find
$$O_{p_2}\{(\mathbf{p}_1+\mathbf{1})-x\,(\mathbf{p}_2)\}=(\mathbf{p}_1+\mathbf{1})-(\mathbf{p}_2),$$
and operating $s+1$ times successively
$$O_{p_2}^{s+1}\{(\mathbf{p}_1+\mathbf{1})-x\,(\mathbf{p}_2)\}=(\mathbf{p}_1+\mathbf{1})-x^{-s}\,(\mathbf{p}_2).$$

It is therefore abundantly clear that two fundamental solutions of the functional equation are
$$(\mathbf{p}_1+\mathbf{1}) \text{ and } (\mathbf{p}_2);$$
and we still obtain a solution if we multiply by any function of x which does not involve p_1 or p_2.

Putting the fundamental solution in evidence we find that we have the expression of $IL\,(p_1p_2;\,\infty)$ given by the formula
$$IL\,(p_1p_2;\,\infty)=\begin{vmatrix}\dfrac{(\mathbf{p}_1+\mathbf{1})}{(\mathbf{1})}, & \dfrac{(\mathbf{p}_2)}{(\mathbf{1})}\\ x, & 1\end{vmatrix}$$
in determinant form.

In the above investigation we have only been concerned with integer values of p_1, p_2 such that $p_1 \geqslant p_2$ because only in that case is $IL\,(p_1p_2;\,\infty)$ an inner lattice function according to definition; but the investigation is equally valid as regards the functional equation when p_1, p_2 are any numerical magnitudes; they need not be integers and there is no necessity for them to be in descending order of magnitude.

If in the functional equation
$$\begin{aligned}&(\mathbf{p}_1+\mathbf{p}_2)\,IL\,(p_1p_2;\,\infty)\\ &=(\mathbf{p}_1+\mathbf{1})\,IL\,(p_1-1\,p_2;\,\infty)+(\mathbf{p}_2)\,IL\,(p_1\,p_2-1;\,\infty)\\ &\quad-(\mathbf{p}_1+\mathbf{1})\,(\mathbf{p}_2)\,IL\,(p_1-1\,p_2-1;\,\infty)\end{aligned}$$
we put $p_2=p_1$ we find
$$\begin{aligned}&(\mathbf{2p}_1)\,IL\,(p_1p_1;\,\infty)\\ &=(\mathbf{p}_1+\mathbf{1})\,IL\,(p_1-1\,p_1;\,\infty)+(\mathbf{p}_1)\,IL\,(p_1p_1-1;\,\infty)\\ &\quad-(\mathbf{p}_1+\mathbf{1})\,(\mathbf{p}_1)\,IL\,(p_1-1\,p_1-1;\,\infty),\end{aligned}$$

wherein $IL(p_1 - 1\, p_1;\, \infty)$ is not an inner lattice function, but still th solution we have obtained is valid. Combining the equation just writtei with

$$(2p_1)\, IL(p_1 p_1;\, \infty) = (p_1)\, IL(p_1\, p_1 - 1;\, \infty),$$

we find

$$IL(p_1 - 1\, p_1;\, \infty) = (p_1)\, IL(p_1 - 1\, p_1 - 1;\, \infty),$$

which is true as a result of the formula

$$IL(p_1 p_2;\, \infty) = \frac{1}{(1)} \{(p_1 + 1) - x\,(p_2)\}.$$

Hence our investigation implies the formula

$$(2p_1)\, IL(p_1 p_1;\, \infty) = (p_1)\, IL(p_1\, p_1 - 1;\, \infty)$$

and we see that

$$IL(p_1 p_1;\, \infty) = \frac{1}{(1)} \{(p_1 + 1) - x\,(p_1)\} = \text{unity}.$$

480. Passing to the Third Order, the functional equation is written in the form

$$\begin{gathered} IL(p_1 p_2 p_3;\, \infty) - (p_2 + 1)\, IL(p_1\, p_2 - 1\, p_3;\, \infty) - (p_3)\, IL(p_1 p_2\, p_3 - 1;\, \infty) \\ + (p_2 + 1)\,(p_3)\, IL(p_1\, p_2 - 1\, p_3 - 1;\, \infty) \\ = x^{p_1 + p_2 + p_3}\, IL(p_1 p_2 p_3;\, \infty) + (p_1 + 2)\, IL(p_1 - 1\, p_2 p_3;\, \infty) \\ - (p_2 + 1)\, IL(p_1 - 1\, p_2 - 1\, p_3;\, \infty) \\ - (p_3)\, IL(p_1 - 1\, p_2\, p_3 - 1;\, \infty) + (p_2 + 1)\,(p_3)\, IL(p_1 - 1\, p_2 - 1\, p_3 - 1;\, \infty). \end{gathered}$$

Writing

$$\begin{gathered} V(p_1 p_2 p_3;\, \infty) = x^{-p_2 - p_3} \{IL(p_1 p_2 p_3;\, \infty) - (p_2 + 1)\, IL(p_1\, p_2 - 1\, p_3;\, \infty) \\ - (p_3)\, IL(p_1 p_2\, p_3 - 1;\, \infty) + (p_2 + 1)\,(p_3)\, IL(p_1 p_2 - 1\, p_3 - 1;\, \infty)\}, \end{gathered}$$

we derive the relation

$$\begin{gathered} (p_1 + p_2 + p_3)\, V(p_1 p_2 p_3;\, \infty) \\ = (p_1 + 2)\, V(p_1 - 1\, p_2 p_3;\, \infty) + (p_2 + 1)\, V(p_1\, p_2 - 1\, p_3;\, \infty) \\ + (p_3)\, V(p_1 p_2\, p_3 - 1;\, \infty) \\ - (p_1 + 2)\,(p_2 + 1)\, V(p_1 - 1\, p_2 - 1\, p_3;\, \infty) - (p_1 + 2)\,(p_3)\, V(p_1 - 1\, p_2\, p_3 - 1;\, \infty) \\ - (p_2 + 1)\,(p_3)\, V(p_1\, p_2 - 1\, p_3 - 1;\, \infty) \\ + (p_1 + 2)\,(p_2 + 1)\,(p_3)\, V(p_1 - 1\, p_2 - 1\, p_3 - 1;\, \infty). \end{gathered}$$

Comparing with the functional equation it is clear that $V(p_1 p_2 p_3;\, \infty)$ is a solution if $IL(p_1 p_2 p_3;\, \infty)$ be one.

481. Proceeding similarly we find six solutions which are exhihited as operations performed upon $IL(p_1 p_2 p_3;\, \infty)$ as follows:

(i)

$$\begin{gathered} O_{p_1}\, IL(p_1 p_2 p_3;\, \infty) \\ = x^{-p_1} \{IL(p_1 p_2 p_3;\, \infty) - (p_1 + 2)\, IL(p_1 - 1\, p_2 p_3;\, \infty)\}; \end{gathered}$$

(ii)
$$O_{p_2} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_2}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_2+\mathbf{1})\, IL(p_1\, p_2-1\, p_3;\infty)\};$$

(iii)
$$O_{p_3} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_3}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_3)\, IL(p_1p_2\, p_3-1;\infty)\};$$

(iv)
$$O_{p_1p_2} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_1-p_2}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_1+\mathbf{2})\, IL(p_1-1\, p_2p_3;\infty)$$
$$- (\mathbf{p}_2+\mathbf{1})\, IL(p_1\, p_2-1\, p_3;\infty) + (\mathbf{p}_1+\mathbf{2})(\mathbf{p}_2+\mathbf{1})\, IL(p_1-1\, p_2-1\, p_3;\infty)\};$$

(v)
$$O_{p_1p_3} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_1-p_3}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_1+\mathbf{2})\, IL(p_1-1\, p_2p_3;\infty)$$
$$- (\mathbf{p}_3)\, IL(p_1p_2\, p_3-1;\infty) + (\mathbf{p}_1+\mathbf{2})(\mathbf{p}_3)\, IL(p_1-1\, p_2\, p_3-1;\infty)\};$$

(vi)
$$O_{p_2p_3} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_2-p_3}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_2+\mathbf{1})\, IL(p_1\, p_2-1\, p_3;\infty)$$
$$- (\mathbf{p}_3)\, IL(p_1p_2\, p_3-1;\infty) + (\mathbf{p}_2+\mathbf{1})(\mathbf{p}_3)\, IL(p_1\, p_2-1\, p_3-1;\infty)\}.$$

To these may be added for the sake of symmetry

$$O_{p_1p_2p_3} IL(p_1p_2p_3;\infty)$$
$$= x^{-p_1-p_2-p_3}\{IL(p_1p_2p_3;\infty) - (\mathbf{p}_1+\mathbf{2})\, IL(p_1-1\, p_2p_3;\infty)$$
$$- (\mathbf{p}_2+\mathbf{1})\, IL(p_1\, p_2-1\, p_3;\infty)$$
$$- (\mathbf{p}_3)\, IL(p_1p_2\, p_3-1;\infty) + (\mathbf{p}_1+\mathbf{2})(\mathbf{p}_2+\mathbf{1})\, IL(p_1-1\, p_2-1\, p_3;\infty)$$
$$+ (\mathbf{p}_1+\mathbf{2})(\mathbf{p}_3)\, IL(p_1-1\, p_2\, p_3-1;\infty) + (\mathbf{p}_2+\mathbf{1})(\mathbf{p}_3)\, IL(p_1\, p_2-1\, p_3-1;\infty)$$
$$- (\mathbf{p}_1+\mathbf{2})(\mathbf{p}_2+\mathbf{1})(\mathbf{p}_3)\, IL(p_1-1\, p_2-1\, p_3-1;\infty)\},$$

and it is easy to establish the operator relations

$$O_{p_1}O_{p_2}O_{p_3} = 1,$$
$$O_{p_1}O_{p_2} = O_{p_1p_2},\quad O_{p_1}O_{p_3} = O_{p_1p_3},\quad O_{p_2}O_{p_3} = O_{p_2p_3},$$
$$O_{p_1}O_{p_2p_3} = O_{p_2}O_{p_1p_3} = O_{p_3}O_{p_1p_2} = O_{p_1p_2p_3} = 1.$$

482. We now perform these operations upon the known solution of the functional equation. Clearing it of fractions, by multiplying throughout by $(\mathbf{1})^2(\mathbf{2})$, this may be written:

$$(\mathbf{1}) - x^{p_1+1}(\mathbf{2})(\mathbf{p}_2+\mathbf{1}) - x^{p_1+2}(\mathbf{p}_1+\mathbf{1}) + x^{p_2+2}(\mathbf{p}_2) + x^{p_1+2p_2+4}(\mathbf{p}_1-\mathbf{p}_2)$$
$$- x\{(\mathbf{2}) + x^{p_2+2}(1+x)(\mathbf{p}_1-\mathbf{p}_2-\mathbf{1}) - x^{2p_2+3}(\mathbf{2p}_1-\mathbf{2p}_2)\}(\mathbf{p}_3)$$
$$+ x^3\{(\mathbf{1}) + x^{p_2+2}(\mathbf{p}_1-\mathbf{p}_2)\}(\mathbf{p}_3-\mathbf{1})(\mathbf{p}_3).$$

Operating s times successively with O_{p_3} we obtain the function

$$(\mathbf{1}) - x^{p_1+1}(\mathbf{2})(\mathbf{p}_2+\mathbf{1}) - x^{p_1+2}(\mathbf{p}_1+\mathbf{1}) + x^{p_2+2}(\mathbf{p}_2) + x^{p_1+2p_2+4}(\mathbf{p}_1-\mathbf{p}_2)$$
$$- x^{s+1}\{(\mathbf{2}) + x^{p_2+2}(1+x)(\mathbf{p}_1-\mathbf{p}_2-\mathbf{1}) - x^{2p_2+3}(\mathbf{2p}_1-\mathbf{2p}_2)\}(\mathbf{p}_3)$$
$$+ x^{-2s+3}\{(\mathbf{1}) + x^{p_2+2}(\mathbf{p}_1-\mathbf{p}_2)\}(\mathbf{p}_3-\mathbf{1})(\mathbf{p}_3),$$

and since s is any integer at pleasure we conclude that the functions

$$P_1 = (\mathbf{1}) - x^{p_1+1}(\mathbf{2})(\mathbf{p_2+1}) - x^{p_1+2}(\mathbf{p_1+1}) + x^{p_2+2}(\mathbf{p_2}) + x^{p_1+2p_2+4}(\mathbf{p_1-p_2}),$$
$$P_2 = \{(\mathbf{2}) + x^{p_2+2}(1+x)(\mathbf{p_1-p_2-1}) - x^{2p_2+3}(\mathbf{2p_1-2p_2})\}(\mathbf{p_3}),$$
$$P_3 = \{(\mathbf{1}) + x^{p_2+2}(\mathbf{p_1-p_2})\}(\mathbf{p_3-1})(\mathbf{p_3}),$$

are solutions of the functional equation.

These results are easily verified by shewing that

$$O_{p_3}P_1 = P_1, \quad O_{p_3}P_2 = x^{-1}P_2, \quad O_{p_3}P_3 = x^{-2}P_3.$$

We now operate with O_{p_2} s times successively upon P_1, when we obtain

$$(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_2+1}) - x^{-s+1}(\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_1+2}),$$

leading to the conclusion that the functions

$$(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_2+1}), \quad (\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_1+2})$$

are solutions of the functional equation, and moreover they are fundamental solutions since no simpler functions can be produced from them by means of the operations.

Again operate with O_{p_2} s times successively upon P_2 and we find

$$x^s(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_3}) - x^{-s+2}(\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_3}),$$

and we conclude that the functions

$$(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_3}), \quad (\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_3})$$

are fundamental solutions.

Finally operating with O_{p_2} upon P_3 we obtain

$$-x(\mathbf{p_3-1})(\mathbf{p_3})(\mathbf{p_2+1}) + x^s(\mathbf{p_3-1})(\mathbf{p_3})(\mathbf{p_1+2}),$$

establishing that the functions

$$(\mathbf{p_3-1})(\mathbf{p_3})(\mathbf{p_2+1}), \quad (\mathbf{p_3-1})(\mathbf{p_3})(\mathbf{p_1+2})$$

are fundamental solutions.

We have thus six fundamental solutions

$$(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_2+1}), \quad (\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_1+2}), \quad (\mathbf{p_3})(\mathbf{p_3-1})(\mathbf{p_1+2}),$$
$$(\mathbf{p_1+1})(\mathbf{p_1+2})(\mathbf{p_3}), \quad (\mathbf{p_2})(\mathbf{p_2+1})(\mathbf{p_3}), \quad (\mathbf{p_3})(\mathbf{p_3-1})(\mathbf{p_2+1}),$$

and no more are obtainable.

483. The known solution of the functional equation, which is the inner lattice function we require, can now be expressed in terms of these. Since it has been found that

$$O_{p_3}^s(\mathbf{1})^2(\mathbf{2})\,IL(p_1p_2p_3;\infty) = P_1 - x^{s+1}P_2 + x^{-2s+3}P_3,$$

by putting $s = 0$ we find

$$(\mathbf{1})^2(\mathbf{2})\,IL(p_1p_2p_3;\infty) = P_1 - xP_2 + x^3P_3;$$

also putting $s=0$ in the results

$$O_{p_2}^s P_1 = (\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_2+1) - x^{-s+1}(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_1+2),$$
$$O_{p_2}^s P_2 = x^s(\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_3) - x^{-s+2}(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_3),$$
$$O_{p_2}^s P_3 = -x(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_2+1) + x^s(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_1+2),$$

it appears that

$$P_1 = (\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_2+1) - x(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_1+2),$$
$$P_2 = (\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_3) - x^2(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_3),$$
$$P_3 = -x(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_2+1) + (\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_1+2);$$

whence, substituting,

$$(1)^2(2)\, IL(p_1p_2p_3;\infty)$$
$$= (\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_2+1) - x(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_1+2) - x(\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_3)$$
$$+ x^3(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_3) + x^3(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_1+2) - x^4(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_2+1),$$

or

$$IL(p_1p_2p_3;\infty) = \begin{vmatrix} \frac{(\mathrm{p}_1+1)}{(1)}\cdot\frac{(\mathrm{p}_1+2)}{(2)} & \frac{(\mathrm{p}_2)}{(1)}\cdot\frac{(\mathrm{p}_2+1)}{(2)} & x\frac{(\mathrm{p}_3-1)}{(1)}\cdot\frac{(\mathrm{p}_3)}{(2)} \\ x\frac{(\mathrm{p}_1+2)}{(1)} & \frac{(\mathrm{p}_2+1)}{(1)} & \frac{(\mathrm{p}_3)}{(1)} \\ x^3 & x & 1 \end{vmatrix},$$

an elegant expression of the inner lattice function in determinant form.

Reasoning as in the case of the second order it is not difficult to see that this expression of the inner lattice function is valid whatever equalities exist between the numbers p_1, p_2, p_3.

It is to be noticed that if it is convenient for any purpose we may take out the factor $\frac{1}{(1)^2(2)}$; and since

$$(1)^2(2) = \begin{vmatrix} 1 & 1 & x \\ x & 1 & 1 \\ x^3 & x & 1 \end{vmatrix},$$

we may express the inner lattice function as the quotient of two determinants.

484. Passing now to the Fourth Order $IL(p_1p_2p_3p_4;\infty)$ and guided by the foregoing results put

$$A_0=1,\quad A_1=(\mathrm{p}_1+3),\quad A_2=(\mathrm{p}_1+2)(\mathrm{p}_1+3),\quad A_3=(\mathrm{p}_1+1)(\mathrm{p}_1+2)(\mathrm{p}_1+3),$$
$$B_0=1,\quad B_1=(\mathrm{p}_2+2),\quad B_2=(\mathrm{p}_2+1)(\mathrm{p}_2+2),\quad B_3=(\mathrm{p}_2)(\mathrm{p}_2+1)(\mathrm{p}_2+2),$$
$$C_0=1,\quad C_1=(\mathrm{p}_3+1),\quad C_2=(\mathrm{p}_3)(\mathrm{p}_3+1),\quad C_3=(\mathrm{p}_3-1)(\mathrm{p}_3)(\mathrm{p}_3+1),$$
$$D_0=1,\quad D_1=(\mathrm{p}_4),\quad D_2=(\mathrm{p}_4-1)(\mathrm{p}_4),\quad D_3=(\mathrm{p}_4-2)(\mathrm{p}_4-1)(\mathrm{p}_4),$$

and consider the twenty-four products

$$\begin{array}{llll}
A_0B_1C_2D_3 & A_1B_0C_2D_3 & A_2B_0C_1D_3 & A_3B_0C_1D_2 \\
A_0B_1C_3D_2 & A_1B_0C_3D_2 & A_2B_0C_3D_1 & A_3B_0C_2D_1 \\
A_0B_2C_1D_3 & A_1B_2C_0D_3 & A_2B_1C_0D_3 & A_3B_1C_0D_2 \\
A_0B_2C_3D_1 & A_1B_2C_3D_0 & A_2B_1C_3D_0 & A_3B_1C_2D_0 \\
A_0B_3C_1D_2 & A_1B_3C_0D_2 & A_2B_3C_0D_1 & A_3B_2C_0D_1 \\
A_0B_3C_2D_1 & A_1B_3C_2D_0 & A_2B_3C_1D_0 & A_3B_2C_1D_0
\end{array}$$

in which every permutation of the numbers 0, 1, 2, 3 is in evidence.

We shall shew that each of these products is a solution of the functional equation

$$x^{\Sigma p} IL(p_1p_2p_3p_4;\infty) = (1-\phi_1)(1-\phi_2)(1-\phi_3)(1-\phi_4) IL(p_1p_2p_3p_4;\infty)$$

where, as above,

$$\phi_s IL(p_1p_2\ldots p_m;\infty) = (\mathbf{p_s + m - s})\, IL(p_1p_2\ldots p_s - 1\ldots p_m;\infty),$$

for if i, j, k, l be any permutation of 0, 1, 2, 3

$$\phi_1 A_iB_jC_kD_l = B_jC_kD_l\phi_1 A_i.$$

Now

$$\phi_1 A_1 = (\mathbf{p_1+4-1})(\mathbf{p_1+2}) = (\mathbf{p_1+2})A_1 = (\mathbf{p_1-1+3})A_1,$$

and in general $\qquad \phi_1 A_i = (\mathbf{p_1 - i + 3}) A_i,$

so that $\qquad \phi_1 A_iB_jC_kD_l = (\mathbf{p_1 - i + 3}) A_iB_jC_kD_l,$

and similarly $\qquad \phi_2 A_iB_jC_kD_l = (\mathbf{p_2 - j + 2}) A_iB_jC_kD_l,$

$$\phi_3 A_iB_jC_kD_l = (\mathbf{p_3 - k + 1}) A_iB_jC_kD_l,$$

$$\phi_4 A_iB_jC_kD_l = (\mathbf{p_4 - l}) A_iB_jC_kD_l.$$

Hence $\qquad (1-\phi_1)(1-\phi_2)(1-\phi_3)(1-\phi_4) A_iB_jC_kD_l$

$$= \{1-(\mathbf{p_1 - i + 3})\}\{1-(\mathbf{p_2 - j + 2})\}\{1-(\mathbf{p_3 - k + 1})\}\{1-(\mathbf{p_4 - l})\} A_iB_jC_kD_l$$

$$= x^{p_1-i+3+p_2-j+2+p_3-k+1+p_4-l} A_iB_jC_kD_l$$

$$= x^{\Sigma p} A_iB_jC_kD_l,$$

since $\qquad i+j+k+l = 0+1+2+3 = 6.$

485. It is thus established that each of the twenty-four products is a solution of the functional equation. Hence the determinant which is a linear function of these products, viz.:

$$\begin{vmatrix} A_3 & B_3 & C_3 & D_3 \\ xA_2 & B_2 & C_2 & xD_2 \\ x^3A_1 & xB_1 & C_1 & D_1 \\ x^6 & x^3 & x & 1 \end{vmatrix},$$

is also a solution; and we shall shew that when divided by $(1)^3 (2)^2 (3)$, which in determinant form is

$$\begin{vmatrix} 1 & 1 & x & x^3 \\ x & 1 & 1 & x \\ x^3 & x & 1 & 1 \\ x^6 & x^3 & x & 1 \end{vmatrix},$$

it is the actual expression of $IL(p_1p_2p_3p_4; \infty)$.

The expression is

$$\begin{vmatrix} \frac{(p_1+1)(p_1+2)(p_1+3)}{(1)(2)(3)}, & \frac{(p_2)(p_2+1)(p_2+2)}{(1)(2)(3)}, & \frac{x(p_3-1)(p_3)(p_3+1)}{(1)(2)(3)}, & \frac{x^3(p_4-2)(p_4-1)(p_4)}{(1)(2)(3)} \\ \frac{x(p_1+2)(p_1+3)}{(1)(2)}, & \frac{(p_2+1)(p_2+2)}{(1)(2)}, & \frac{(p_3)(p_3+1)}{(1)(2)}, & \frac{x(p_4-1)(p_4)}{(1)(2)} \\ \frac{x^3(p_1+3)}{(1)}, & \frac{x(p_2+2)}{(1)}, & \frac{(p_3+1)}{(1)}, & \frac{(p_4)}{(1)} \\ x^6, & x^3, & x, & 1 \end{vmatrix}$$

486. We first take the test of the sum of the coefficients.

Putting $x=1$ it becomes

$$\frac{1}{12}\begin{vmatrix} (p_1+1)(p_1+2)(p_1+3), & p_2(p_2+1)(p_2+2), & (p_3-1)(p_3)(p_3+1), & (p_4-2)(p_4-1)(p_4) \\ (p_1+2)(p_1+3), & (p_2+1)(p_2+2), & (p_3)(p_3+1), & (p_4-1)(p_4) \\ (p_1+3), & (p_2+2), & (p_3+1), & (p_4) \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

wherein of course (p_1+1) denotes the number p_1+1 and not $1-x^{p_1+1}$.

Now if, herein, we put separately

$$p_1-p_2+1,\quad p_1-p_3+2,\quad p_1-p_4+3,\quad p_2-p_3+1,\quad p_2-p_4+2,\quad p_3-p_4+1$$

equal to zero, we find that in each case two columns become identical and the determinant vanishes. Each of these numbers is therefore a factor of the determinant and the expression is easily seen to be

$$\tfrac{1}{12}(p_1-p_2+1)(p_1-p_3+2)(p_1-p_4+3)(p_2-p_3+1)(p_2-p_4+2)(p_3-p_4+1)$$

which we know otherwise to be the sum of the coefficients in the inner lattice function.

487. Also on putting $p_1 = p_2 = p_3 = p_4 = p$, the function ought to become equal to unity. Apart from the factor $\frac{1}{(1)^3\,(2)^2\,(3)}$ the expression becomes

$$\left|\begin{array}{cccc} (p+1)(p+2)(p+3), & (p)(p+1)(p+2), & x(p-1)(p)(p+1), & x^3(p-2)(p-1)(p) \\ x(p+2)(p+3), & (p+1)(p+2), & (p)(p+1), & x(p-1)(p) \\ x^3(p+3), & x(p+2), & (p+1), & (p) \\ x^6, & x^3, & x, & 1 \end{array}\right|$$

Transform this by taking—

For New First Row:

$$\text{1st Row} + x^p\frac{(3)}{(1)} \times \text{2nd Row} + x^{2p+1}\frac{(3)}{(1)} \times \text{3rd Row} + x^{3p+3} \times \text{4th Row};$$

For New Second Row:

$$\text{2nd Row} + x^p\frac{(2)}{(1)} \times \text{3rd Row} + x^{2p+1} \times \text{4th Row};$$

For New Third Row:

$$\text{3rd Row} + x^p \times \text{4th Row};$$

and it becomes

$$\left|\begin{array}{cccc} 1 & 1 & x & x^3 \\ x & 1 & 1 & x \\ x^3 & x & 1 & 1 \\ x^6 & x^3 & x & 1 \end{array}\right|,$$

which is an expression of $(1)^3\,(2)^2\,(3)$.

Hence the value of the determinant expression for $IL\,(pppp;\,\infty)$ is unity.

It can also be shewn that putting $p_1 = p_2 = p_3 = p$, $p_4 = p-1$ makes the expression equal to $1 + x^p + x^{2p} + x^{3p}$ and this can be verified in particular cases.

488. All of the above processes are valid when applied to the functional equation of order m and lead to a determinant expression of $IL\,(p_1 p_2 \ldots p_m;\,\infty)$. The constituent of the determinant which is in the rth row and cth column apart from the power of x which appears explicitly is

$$\frac{(p_c + r - c + 1)\,(p_c + r - c + 2)\ldots(p_c + m - c)}{(1)\,(2)\ldots(m-r)}.$$

The part which involves x explicitly presents exponents of x which are figurate numbers of the third order.

It is exhibited in the determinant

$$\begin{vmatrix} 1 & 1 & x & x^3 & \dots x^{\binom{m-1}{2}} \\ x & 1 & 1 & x & \dots x^{\binom{m-}{2}} \\ x^3 & x & 1 & 1 & \dots x^{\binom{m-3}{2}} \\ \vdots & \vdots & \vdots & \vdots & \dots \quad \vdots \\ x^{\binom{m}{2}} & x^{\binom{m-1}{2}} & x^{\binom{m-2}{2}} & x^{\binom{m-3}{2}} \dots & 1 \end{vmatrix}.$$

489. The results obtained for the second and third orders are now set forth:

$$IL(p_1p_2;\ \infty) = \begin{vmatrix} \dfrac{(\mathrm{p}_1+1)}{(1)} & \dfrac{(\mathrm{p}_2)}{(1)} \\ x & 1 \end{vmatrix},$$

$$L(p_1p_2;\ \infty) = \frac{(1)\,(2)\dots(\mathrm{p}_1+\mathrm{p}_2)}{(1)\,(2)\dots(\mathrm{p}_1+1)\,.\,(1)\,(2)\dots(\mathrm{p}_2)} \cdot \begin{vmatrix} (\mathrm{p}_1+1) & (\mathrm{p}_2) \\ x & 1 \end{vmatrix},$$

$$GF(p_1p_2;\ \infty) = \frac{1}{(1)\,(2)\dots(\mathrm{p}_1+1)\,.\,(1)\,(2)\dots(\mathrm{p}_2)} \cdot \begin{vmatrix} (\mathrm{p}_1+1) & (\mathrm{p}_2) \\ x & 1 \end{vmatrix},$$

$$IL(p_1p_2p_3;\ \infty) = \begin{vmatrix} \dfrac{(\mathrm{p}_1+1)\,(\mathrm{p}_1+2)}{(1)\,(2)} & \dfrac{(\mathrm{p}_2)\,(\mathrm{p}_2+1)}{(1)\,(2)} & x\dfrac{(\mathrm{p}_3-1)\,(\mathrm{p}_3)}{(1)\,(2)} \\ x\dfrac{(\mathrm{p}_1+2)}{(1)} & \dfrac{(\mathrm{p}_2+1)}{(1)} & \dfrac{(\mathrm{p}_3)}{(1)} \\ x^3 & x & 1 \end{vmatrix},$$

$$L(p_1p_2p_3;\ \infty) = \frac{(1)\,(2)\dots(\mathrm{p}_1+\mathrm{p}_2+\mathrm{p}_3)}{(1)\,(2)\dots(\mathrm{p}_1+2)\,.\,(1)\,(2)\dots(\mathrm{p}_2+1)\,.\,(1)\,(2)\dots(\mathrm{p}_3)}$$

$$\times \begin{vmatrix} (\mathrm{p}_1+1)\,(\mathrm{p}_1+2) & (\mathrm{p}_2)\,(\mathrm{p}_2+1) & x\,(\mathrm{p}_3-1)\,(\mathrm{p}_3) \\ x\,(\mathrm{p}_1+2) & (\mathrm{p}_2+1) & (\mathrm{p}_3) \\ x^3 & x & 1 \end{vmatrix},$$

$$GF(p_1p_2p_3;\ \infty) = \frac{1}{(1)\,(2)\dots(\mathrm{p}_1+2)\,.\,(1)\,(2)\dots(\mathrm{p}_2+1)\,.\,(1)\,(2)\dots(\mathrm{p}_3)}$$

$$\times \begin{vmatrix} (\mathrm{p}_1+1)\,(\mathrm{p}_1+2) & (\mathrm{p}_2)\,(\mathrm{p}_2+1) & x\,(\mathrm{p}_3-1)\,(\mathrm{p}_3) \\ x\,(\mathrm{p}_1+2) & (\mathrm{p}_2+1) & (\mathrm{p}_3) \\ x^3 & x & 1 \end{vmatrix}.$$

The results for the fourth order are similarly written down.

In particular $\qquad GF(p^m;\ \infty)$

$$= \frac{1}{(\mathrm{m})\,(\mathrm{m}+1)\dots(\mathrm{p}+\mathrm{m}-1)\,.\,(\mathrm{m}-1)\,(\mathrm{m})\dots(\mathrm{p}+\mathrm{m}-2)\dots(1)\,(2)\dots(\mathrm{p})},$$

the inner lattice function being unity.

CHAPTER II

PLANE PARTITIONS WITH RESTRICTED PART MAGNITUDE

490. When the part magnitude is restricted not to exceed the number n we have the formulæ

$$GF(p_1p_2\dots p_m;\,n)=\frac{L(p_1p_2\dots p_m;\,n)}{(1)\,(2)\dots(\Sigma p)},$$

$$L(p_1p_2\dots p_m;\,n)=(n+1)\,(n+2)\dots(n+\Sigma p)\,L_0$$
$$+\,(n)\,(n+1)\dots(n+\Sigma p-1)\,L_1+\dots,$$

wherein for brevity $L_s(p_1p_2\dots p_m;\,\infty)$, the sub-lattice function of order s, is written L_s.

The inner lattice function, for a restricted part magnitude, is defined by the relation

$$L(p_1p_2\dots p_m;\,n)$$
$$=\frac{(n+m)\,(n+m+1)\dots(n+p_1+m-1)\,.\,(n+m-1)\dots(n+p_2+m-2)\dots\dots(n+1)\dots(n+p_m)}{(m)\,(m+1)\dots(p_1+m-1)\,.\,(m-1)\,(m)\dots(p_2+m-2)\dots\dots(1)\,(2)\dots(p_m)}$$
$$\times\frac{IL(p_1p_2\dots p_m;\,n)}{(1)\,(2)\dots(\Sigma p)}$$
$$=OL(p_1p_2\dots p_m;\,n)\,.\,IL(p_1p_2\dots p_m;\,n),$$

where the outer lattice function, for a restricted part magnitude, is of fixed form, and we have before us the inner lattice for investigation.

We are guided in fixing the form of the outer function by analogy with the case $n=\infty$. We have not the sum of the coefficients in $L(p_1p_2\dots p_m;\,n)$ before us.

491. We take as the point of departure the functional equation of Art. 471,

$$(1-\theta_1)\,(1-\theta_2)\dots(1-\theta_m)\,GF(p_1p_2\dots p_m;\,n)=x^{\Sigma p}\,GF(p_1p_2\dots p_m;\,n-1),$$

where θ_s is a symbol which operates as explained above.

Converting this into a functional equation for the lattice function we find for the order two

$$L(p_1p_2;n) - x^{p_1+p_2} L(p_1p_2;n-1)$$
$$= (\mathbf{p_1+p_2})\{L(p_1-1\,p_2;n) + L(p_1\,p_2-1;n)\}$$
$$- (\mathbf{p_1+p_2-1})(\mathbf{p_1+p_2})\,L(p_1-1\,p_2-1;n),$$

and for the order three

$$L(p_1p_2p_3;n) - x^{p_1+p_2+p_3} L(p_1p_2p_3;n-1)$$
$$= (\mathbf{p_1+p_2+p_3})\{L(p_1-1\,p_2p_3;n) + L(p_1\,p_2-1\,p_3;n) + L(p_1p_2\,p_3-1;n)\}$$
$$- (\mathbf{p_1+p_2+p_3-1})(\mathbf{p_1+p_2+p_3})\{L(p_1-1\,p_2-1\,p_3;n) + L(p_1-1\,p_2p_3-1;n)$$
$$+ L(p_1\,p_2-1\,p_3-1;n)\}$$
$$+ (\mathbf{p_1+p_2+p_3-2})(\mathbf{p_1+p_2+p_3-1})(\mathbf{p_1+p_2+p_3})\,L(p_1-1\,p_2-1\,p_3-1;n),$$

and in general for the order m

$$(1-\theta_1X)(1-\theta_2X)\dots(1-\theta_mX)\,L(p_1p_2\dots p_m;n) = x^{\Sigma p} L(p_1p_2\dots p_m;n-1),$$

where symbolically

$$X^s = (\mathbf{\Sigma p})(\mathbf{\Sigma p-1})\dots(\mathbf{\Sigma p-s+1}).$$

Also passing to the inner lattice function we have, for the order two,

$$x^{p_1+p_2}(\mathbf{n})(\mathbf{n+1})\,IL(p_1p_2;n-1)$$
$$= (\mathbf{n+p_1+1})(\mathbf{n+p_2})\,IL(p_1p_2;n) - (\mathbf{p_1+1})(\mathbf{n+p_2})\,IL(p_1-1\,p_2;n)$$
$$- (\mathbf{n+p_1+1})(\mathbf{p_2})\,IL(p_1p_2-1;n) + (\mathbf{p_1+1})(\mathbf{p_2})\,IL(p_1-1\,p_2-1;n).$$

For the order three

$$x^{p_1+p_2+p_3}(\mathbf{n})(\mathbf{n+1})(\mathbf{n+2})\,IL(p_1p_2p_3;n-1)$$
$$= (\mathbf{n+p_1+2})(\mathbf{n+p_2+1})(\mathbf{n+p_3})\,IL(p_1p_2p_3;n)$$
$$- (\mathbf{p_1+2})(\mathbf{n+p_2+1})(\mathbf{n+p_3})\,IL(p_1-1\,p_2p_3;n)$$
$$- (\mathbf{n+p_1+2})(\mathbf{p_2+1})(\mathbf{n+p_3})\,IL(p_1\,p_2-1\,p_3;n)$$
$$- (\mathbf{n+p_1+2})(\mathbf{n+p_2+1})(\mathbf{p_3})\,IL(p_1p_2\,p_3-1;n)$$
$$+ (\mathbf{p_1+2})(\mathbf{p_2+1})(\mathbf{n+p_3})\,IL(p_1-1\,p_2-1\,p_3;n)$$
$$+ (\mathbf{p_1+2})(\mathbf{n+p_2+1})(\mathbf{p_3})\,IL(p_1-1\,p_2\,p_3-1;n)$$
$$+ (\mathbf{n+p_1+2})(\mathbf{p_2+1})(\mathbf{p_3})\,IL(p_1\,p_2-1\,p_3-1;n)$$
$$- (\mathbf{p_1+2})(\mathbf{p_2+1})(\mathbf{p_3})\,IL(p_1-1\,p_2-1\,p_3-1;n).$$

In general, if χ_s be a symbol such that

$$\chi_s\,IL(p_1p_2\dots p_m;n) = \frac{(\mathbf{p_s+m-s})}{(\mathbf{n+p_s+m-s})}\,IL(p_1p_2\dots p_s-1\dots p_m;n),$$

then $$x^{\Sigma p}(\mathbf{n})(\mathbf{n+1})\dots(\mathbf{n+m-1})\,IL(p_1p_2\dots p_m;n-1)$$
$$= (\mathbf{n+p_1+m-1})(\mathbf{n+p_2+m-2})\dots(\mathbf{n+p_m})\{(1-\chi_1)(1-\chi_2)\dots(1-\chi_m)\}$$
$$\times IL(p_1p_2\dots p_m;n).$$

492. Taking first the order two, it is convenient to examine several particular cases in order to find the probable form of the inner lattice function. Doing this, the conclusion is that the function is probably

$$1 + x^{p_2+1} \frac{(\mathrm{p}_1 - \mathrm{p}_2)(\mathrm{n})}{(1)(\mathrm{n} + \mathrm{p}_1 + 1)}.$$

Observe that this reduces to the right expression when $n = \infty$. Also it is found to satisfy the functional equation. Moreover, it may be written

$$IL(p_1p_2; n) = \frac{1}{(1)(\mathrm{n} + \mathrm{p}_1 + 1)} \begin{vmatrix} (\mathrm{p}_1 + 1) & (\mathrm{p}_2) \\ x(\mathrm{n}) & (\mathrm{n} + 1) \end{vmatrix},$$

which, in the light of previous results, is very suggestive of truth.

To obtain the fundamental solutions we recall the case of $n = \infty$, and seek solutions of the form

$$\frac{(\mathrm{p}_1 + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n, \quad \frac{(\mathrm{p}_2)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n,$$

where F_n is a function of n to be determined in each case.

We substitute $\frac{(\mathrm{p}_1 + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n$ for $IL(p_1p_2; n)$ in the functional equation and find

$$x^{p_1+p_2}(\mathrm{n})(\mathrm{n} + 1) \frac{(\mathrm{p}_1 + 1)}{(\mathrm{n} + \mathrm{p}_1)} F_{n-1}$$

$$= (\mathrm{n} + \mathrm{p}_1 + 1)(\mathrm{n} + \mathrm{p}_2) \frac{(\mathrm{p}_1 + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n - (\mathrm{p}_1 + 1)(\mathrm{n} + \mathrm{p}_2) \frac{(\mathrm{p}_1)}{(\mathrm{n} + \mathrm{p}_1)} F_n$$

$$- (\mathrm{n} + \mathrm{p}_1 + 1)(\mathrm{p}_2) \frac{(\mathrm{p}_1 + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n + (\mathrm{p}_1 + 1)(\mathrm{p}_2) \frac{(\mathrm{p}_1)}{(\mathrm{n} + \mathrm{p}_1)} F_n,$$

which is equivalent to $\quad (\mathrm{n}) F_n = (\mathrm{n} + 1) F_{n-1},$

from which we deduce $F_n = (\mathrm{n} + 1)$, and we have a fundamental solution

$$\frac{(\mathrm{p}_1 + 1)(\mathrm{n} + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)}.$$

Similarly, when we substitute $\frac{(\mathrm{p}_2)}{(\mathrm{n} + \mathrm{p}_1 + 1)} F_n$ in the functional equation, we find

$$(\mathrm{n} - 1) F_n = (\mathrm{n}) F_{n-1},$$

giving us $F_n = (\mathrm{n})$ and the fundamental solution

$$\frac{(\mathrm{p}_2)(\mathrm{n})}{(\mathrm{n} + \mathrm{p}_1 + 1)},$$

and in terms of these we find

$$IL(p_1p_2; n) = \frac{1}{(1)} \left\{ \frac{(\mathrm{p}_1 + 1)(\mathrm{n} + 1)}{(\mathrm{n} + \mathrm{p}_1 + 1)} - x \frac{(\mathrm{p}_2)(\mathrm{n})}{(\mathrm{n} + \mathrm{p}_1 + 1)} \right\}.$$

493. This simple exposition for the order two points out the path of investigation for the order three; for, guided by the six fundamental solutions when $n=\infty$, it is natural to seek for solutions of the functional equation of the six types:

$$\frac{(p_1+1)(p_1+2)(p_2+1)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,\quad \frac{(p_1+1)(p_1+2)(p_3)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,$$

$$\frac{(p_2)(p_2+1)(p_1+2)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,\quad \frac{(p_2)(p_2+1)(p_3)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,$$

$$\frac{(p_3-1)(p_3)(p_1+2)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,\quad \frac{(p_3-1)(p_3)(p_2+1)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)}F_n,$$

where F_n is a function of n to be determined in *each* case.

Substituting these successively for $IL(p_1p_2p_3;\ n)$ in the equation we find for the six cases respectively:

$$F_n=\frac{(n+1)(n+2)}{(n)^2}F_{n-1};\qquad F_n=\frac{(n+2)}{(n-1)}F_{n-1};$$

$$F_n=\frac{(n+2)}{(n-1)}F_{n-1};\qquad F_n=\frac{(n)(n+2)}{(n-2)(n+1)}F_{n-1};$$

$$F_n=\frac{(n)(n+1)}{(n-1)^2}F_{n-1};\qquad F_n=\frac{(n+1)}{(n-2)}F_{n-1}.$$

The solution of the functional equation

$$F_n=f_nF_{n-1}$$

is clearly

$$F_n=f_nf_{n-1}f_{n-2}\dots,$$

so that if

$$f_n=\frac{(n+1)(n+2)}{(n)^2},$$

then

$$F_n=\frac{(n+1)(n+2)}{(n)^2}\cdot\frac{(n)(n+1)}{(n-1)^2}\cdot\frac{(n-1)(n)}{(n-2)^2}\cdot\frac{(n-2)(n-1)}{(n-3)^2}\dots=(n+1)^2(n+2)$$

and similarly; so that we have the six results:

$$F_n=(n+1)^2(n+2),\qquad F_n=(n)(n+1)(n+2),$$

$$F_n=(n)(n+1)(n+2),\qquad F_n=(n-1)(n)(n+2),$$

$$F_n=(n)^2(n+1),\qquad F_n=(n-1)(n)(n+1),$$

respectively.

We are thus led to the six fundamental solutions:

$$\text{(i)}\quad \frac{(p_1+1)(p_1+2)(p_2+1)(n+1)^2(n+2)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)},$$

$$\text{(ii)}\quad \frac{(p_1+1)(p_1+2)(p_3)(n)(n+1)(n+2)}{(n+p_1+1)(n+p_1+2)(n+p_2+1)},$$

(iii) $\dfrac{(p_2)\,(p_2+1)\,(p_1+2)\,(n)\,(n+1)\,(n+2)}{(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1)}$,

(iv) $\dfrac{(p_2)\,(p_2+1)\,(p_3)\,(n-1)\,(n)\,(n+2)}{(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1)}$,

(v) $\dfrac{(p_3-1)\,(p_3)\,(p_1+2)\,(n)^2\,(n+1)}{(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1)}$,

(vi) $\dfrac{(p_3-1)\,(p_3)\,(p_2+1)\,(n-1)\,(n)\,(n+1)}{(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1)}$.

Guided by previous work we form $IL(p_1p_2p_3;\ n)$ by taking a fraction with numerator

$$\begin{aligned}&(p_1+1)\,(p_1+2)\,(p_2+1)\,(n+1)^2\,(n+2)\\ &-x\,(p_2)\,(p_2+1)\,(p_1+2)\,(n)\,(n+1)\,(n+2)\\ &-x\,(p_1+1)\,(p_1+2)\,(p_3)\,(n)\,(n+1)\,(n+2)\\ &+x^3\,(p_3-1)\,(p_3)\,(p_1+2)\,(n)^2\,(n+1)\\ &+x^3\,(p_2)\,(p_2+1)\,(p_3)\,(n-1)\,(n)\,(n+2)\\ &-x^4\,(p_3-1)\,(p_3)\,(p_2+1)\,(n-1)\,(n)\,(n+1),\end{aligned}$$

and with denominator

$$(1)^2\,(2)\,(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1),$$

and we find that we may write

$$(1)^2\,(2)\,(n+p_1+1)\,(n+p_1+2)\,(n+p_2+1)\,IL(p_1p_2p_3;\ n)$$
$$=\begin{vmatrix}(p_1+1)\,(p_1+2) & (p_2)\,(p_2+1) & x\,(p_3-1)\,(p_3)\\ x\,(p_1+2)\,(n) & (p_2+1)\,(n+1) & (p_3)\,(n+2)\\ x^3\,(n-1)\,(n) & x\,(n)\,(n+1) & (n+1)\,(n+2)\end{vmatrix},$$

an elegant result which suggests the general formula.

494. We may also take the following view:

Recalling a previous notation for the order four

$$A_3=(p_1+1)\,(p_1+2)\,(p_1+3), \text{ etc.}$$

and taking a product

$$A_iB_jC_kD_l\,.\,F_n,$$

where i, j, k, l denote some permutation of the numbers 3, 2, 1, 0 and F_n is a function of n to be determined, we substitute

$$\frac{A_iB_jC_kD_l\,.\,F_n}{(n+p_1+1)\,(n+p_1+2)\,(n+p_1+3)\,.\,(n+p_2+1)\,(n+p_2+2)\,.\,(n+p_3+1)}$$

for $IL(p_1p_2p_3p_4;\ n)$ in the functional equation and find a result which on reduction is

$$F_n = \frac{(\mathbf{n})\,(\mathbf{n}+1)\,(\mathbf{n}+2)\,(\mathbf{n}+3)}{(\mathbf{n}+\mathbf{i}-3)\,(\mathbf{n}+\mathbf{j}-2)\,(\mathbf{n}+\mathbf{k}-1)\,(\mathbf{n}+\mathbf{l})}\,F_{n-1} = f_n F_{n-1},$$

so that $F_n = f_n f_{n-1} f_{n-2} \dots$.

Now write

$$\frac{(\mathbf{n})}{(\mathbf{n}+\mathbf{i}-3)}\cdot\frac{(\mathbf{n}-1)}{(\mathbf{n}+\mathbf{i}-4)}\cdot\frac{(\mathbf{n}-2)}{(\mathbf{n}+\mathbf{i}-5)}\dots\dots = \phi_{ai},$$

$$\frac{(\mathbf{n}+1)}{(\mathbf{n}+\mathbf{j}-2)}\cdot\frac{(\mathbf{n})}{(\mathbf{n}+\mathbf{j}-3)}\cdot\frac{(\mathbf{n}-1)}{(\mathbf{n}+\mathbf{j}-4)}\dots\dots = \phi_{bj},$$

$$\frac{(\mathbf{n}+2)}{(\mathbf{n}+\mathbf{k}-1)}\cdot\frac{(\mathbf{n}+1)}{(\mathbf{n}+\mathbf{k}-2)}\cdot\frac{(\mathbf{n})}{(\mathbf{n}+\mathbf{k}-3)}\dots\dots = \phi_{ck},$$

$$\frac{(\mathbf{n}+3)}{(\mathbf{n}+\mathbf{l})}\cdot\frac{(\mathbf{n}+2)}{(\mathbf{n}+\mathbf{l}-1)}\cdot\frac{(\mathbf{n}+1)}{(\mathbf{n}+\mathbf{l}-2)}\dots\dots = \phi_{dl},$$

$$F_n = \phi_{ai}\,\phi_{bj}\,\phi_{ck}\,\phi_{dl}.$$

We then have

$$(1)^3\,(2)^2\,(3)\,(\mathbf{n}+\mathbf{p}_1+1)\,(\mathbf{n}+\mathbf{p}_1+2)\,(\mathbf{n}+\mathbf{p}_1+3)\,.\,(\mathbf{n}+\mathbf{p}_2+1)\,(\mathbf{n}+\mathbf{p}_2+2)$$
$$\times\,(\mathbf{n}+\mathbf{p}_3+1)\,IL(p_1p_2p_3p_4;\ n)$$
$$= \begin{vmatrix} A_3\phi_{a3} & B_3\phi_{b3} & xC_3\phi_{c3} & x^3D_3\phi_{d3} \\ x\,A_2\phi_{a2} & B_2\phi_{b2} & C_2\phi_{c2} & x\,D_2\phi_{d2} \\ x^3A_1\phi_{a1} & x\,B_1\phi_{b1} & C_1\phi_{c1} & D_1\phi_{d1} \\ x^6A_0\phi_{a0} & x^3B_0\phi_{b0} & xC_0\phi_{c0} & D_0\phi_{d0} \end{vmatrix},$$

where

$\phi_{a3} = \phi_{b3} = \phi_{c3} = \phi_{d3} = 1,$

$\phi_{a2} = (\mathbf{n}),\quad \phi_{b2} = (\mathbf{n}+1),\quad \phi_{c2} = (\mathbf{n}+2),\quad \phi_{d2} = (\mathbf{n}+3),$

$\phi_{a1} = (\mathbf{n}-1)\,(\mathbf{n}),\quad \phi_{b1} = (\mathbf{n})\,(\mathbf{n}+1),\quad \phi_{c1} = (\mathbf{n}+1)\,(\mathbf{n}+2),\quad \phi_{d1} = (\mathbf{n}+2)\,(\mathbf{n}+3)$

$\phi_{a0} = (\mathbf{n}-2)\,(\mathbf{n}-1)\,(\mathbf{n}),\qquad \phi_{b0} = (\mathbf{n}-1)\,(\mathbf{n})\,(\mathbf{n}+1),$

$\phi_{c0} = (\mathbf{n})\,(\mathbf{n}+1)\,(\mathbf{n}+2),\qquad \phi_{d0} = (\mathbf{n}+1)\,(\mathbf{n}+2)\,(\mathbf{n}+3),$

so that the determinant is

$$\begin{vmatrix} (\mathbf{p}_1+1)(\mathbf{p}_1+2)(\mathbf{p}_1+3) & (\mathbf{p}_2)\,(\mathbf{p}_2+1)\,(\mathbf{p}_2+2) & x(\mathbf{p}_3-1)\,(\mathbf{p}_3)\,(\mathbf{p}_3+1) & x^3(\mathbf{p}_4-2)\,(\mathbf{p}_4-1 \\ x\,(\mathbf{n})\,(\mathbf{p}_1+2)\,(\mathbf{p}_1+3) & (\mathbf{n}+1)(\mathbf{p}_2+1)(\mathbf{p}_2+2) & (\mathbf{n}+2)\,(\mathbf{p}_3)\,(\mathbf{p}_3+1) & x\,(\mathbf{n}+3)\,(\mathbf{p}_4-1) \\ x^3\,(\mathbf{n}-1)\,(\mathbf{n})\,(\mathbf{p}_1+3) & x\,(\mathbf{n})\,(\mathbf{n}+1)\,(\mathbf{p}_2+2) & (\mathbf{n}+1)(\mathbf{n}+2)(\mathbf{p}_3+1) & (\mathbf{n}+2)\,(\mathbf{n}+3)(\\ x^6\,(\mathbf{n}-2)\,(\mathbf{n}-1)\,(\mathbf{n}) & x^3\,(\mathbf{n}-1)\,(\mathbf{n})\,(\mathbf{n}+1) & x\,(\mathbf{n})\,(\mathbf{n}+1)\,(\mathbf{n}+2) & (\mathbf{n}+1)\,(\mathbf{n}+2)\,(\mathbf{n} \end{vmatrix}$$

This is evidently a general process and suffices to establish that a solution of the functional equation of order m is a determinant of order m in which, to a power of x *près*, the constituent in the rth row and cth column is

$$(\mathrm{p_c}+\mathrm{r}-\mathrm{c}+1)\ldots(\mathrm{p_c}+\mathrm{m}-\mathrm{c})\,.\,(\mathrm{n}-\mathrm{r}+\mathrm{c}+1)\ldots(\mathrm{n}+\mathrm{c}-1),$$

the associated power of x being $x^{\binom{r-c+1}{2}}$ when the constituent is below the principal diagonal and $x^{\binom{c-r}{2}}$ when above; the determinant being divided by

$$(1)^{m-1}(2)^{m-2}\ldots(\mathrm{m}-1)(\mathrm{n}+\mathrm{p}_1+1)\ldots(\mathrm{n}+\mathrm{p}_1+\mathrm{m}-1)$$
$$\times(\mathrm{n}+\mathrm{p}_2+1)\ldots(\mathrm{n}+\mathrm{p}_2+\mathrm{m}-2)\ldots\ldots(\mathrm{n}+\mathrm{p}_{\mathrm{m}-1}+1).$$

This is the expression of $IL(p_1p_2\ldots p_m;\ n)$.

It clearly reduces to $IL(p_1p_2\ldots p_m;\ \infty)$ as previously found when $n=\infty$.

To illustrate the properties of the determinant we will consider either the determinant of order three or that of order four.

If we put $p_1=p_2=\ldots=p_m=l$, the expression of $IL(p_1p_2\ldots p_m;\ n)$ should become unity. It is then in fact

$$IL(l;\ m;\ n),$$

and is concerned with a lattice in which the nodes in the directions of the axes x, y, z do not exceed l, m, n in number respectively.

$GF(l;\ m;\ n)$, $L(l;\ m;\ n)$ and $IL(l;\ m;\ n)$ are each of them symmetrical as regards the numbers l, m, n. In particular the determinant portion of $IL(l;\ m;\ n)$ is symmetrical in the numbers l, n.

In the determinant of order three put $p_1=p_2=p_3=l$, and we find

$$\begin{vmatrix} (\mathrm{l}+1)(\mathrm{l}+2) & (\mathrm{l})(\mathrm{l}+1) & x(\mathrm{l}-1)(\mathrm{l}) \\ x(\mathrm{n})(\mathrm{l}+2) & (\mathrm{n}+1)(\mathrm{l}+1) & (\mathrm{n}+2)(\mathrm{l}) \\ x^3(\mathrm{n}-1)(\mathrm{n}) & x(\mathrm{n})(\mathrm{n}+1) & (\mathrm{n}+1)(\mathrm{n}+2) \end{vmatrix},$$

which must be symmetrical in l, n.

The associated denominator is

$$(1)^2(2)(\mathrm{n}+\mathrm{l}+1)^2(\mathrm{n}+\mathrm{l}+2),$$

and we have to shew that the determinant is equal to it.

If we put $n=\infty$, the determinant becomes $(1)^2(2)$; that is to say, n and l disappear together. It follows from this remark that the determinant must be a function of $n+l$. We therefore put $l=0$ in the determinant in order to find it as a function of n. If then we substitute $n+l$ for n, we must get the expression of the determinant.

For example, putting $l=0$, the determinant becomes

$$\begin{vmatrix} (1)\,(2) & 0 & 0 \\ x\,(n)\,(2) & (n+1)\,(1) & 0 \\ x^3\,(n-1)\,(n) & x\,(n)\,(n+1) & (n+1)\,(n+2) \end{vmatrix},$$

which is $(1)^2\,(2)\,(n+1)^2\,(n+2)$.

Thence writing $n+l$ for n, we see that the former determinant is

$$(1)^2\,(2)\,(n+l+1)^2\,(n+l+2),$$

and thus
$$IL\,(l\,;\,3\,;\,n)=1.$$

The reasoning is general, and establishes the result

$$IL\,(l\,;\,m\,;\,n)=1.$$

This may be verified in numerous special cases, and suffices to confirm the expression obtained for

$$IL\,(p_1p_2\ldots p_m\,;\,n).$$

495. The results that have been established may now be set forth as follows:

$$IL\,(p_1p_2\,;\,n)=\frac{1}{(1)\,(n+p_1+1)}\begin{vmatrix} (p_1+1) & (p_2) \\ x\,(n) & (n+1) \end{vmatrix};$$

$$L\,(p_1p_2\,;\,n)=(1)\,(2)\ldots(p_1+p_2)\,.\,\frac{(n+2)\ldots(n+p_1+1)\,.\,(n+1)\ldots(n+p_2)}{(2)\ldots(p_1+1)\,.\,(1)\ldots(p_2)}$$
$$\times\frac{1}{(1)\,(n+p_1+1)}\begin{vmatrix} (p_1+1) & (p_2) \\ x\,(n) & (n+1) \end{vmatrix};$$

$$GF\,(p_1p_2\,;\,n)=\frac{(n+2)\ldots(n+p_1+1)\,.\,(n+1)\ldots(n+p_2)}{(2)\ldots(p_1+1)\,.\,(1)\ldots(p_2)}$$
$$\times\frac{1}{(1)\,(n+p_1+1)}\begin{vmatrix} (p_1+1) & (p_2) \\ x\,(n) & (n+1) \end{vmatrix};$$

$$IL\,(ll\,;\,n)=IL\,(l\,;\,2\,;\,n)=1\,;$$

$$L\,(ll\,;\,n)=L\,(l\,;\,2\,;\,n)=(1)\,(2)\ldots(2l)\,.\,\frac{(n+2)\ldots(n+l+1)\,.\,(n+1)\ldots(n+l)}{(2)\ldots(l+1)\,.\,(1)\ldots(l)}\,;$$

$$GF\,(ll\,;\,n)=GF\,(l\,;\,2\,;\,n)=\frac{(n+2)\ldots(n+l+1)\,.\,(n+1)\ldots(n+l)}{(2)\ldots(l+1)\,.\,(1)\ldots(l)}.$$

Observe that we may also write

$$GF\,(p_1p_2\,;\,n)=\frac{(n+2)\ldots(n+p_1)\,.\,(n+1)\ldots(n+p_2)}{(1)\ldots(p_1+1)\,.\,(1)\ldots(p_2)}\begin{vmatrix} (p_1+1) & (p_2) \\ x\,(n) & (n+1) \end{vmatrix},$$

a form which has some advantage.

We have yet another form, viz.:

$$GF(p_1p_2;\, n) = \frac{(1)\ldots(n+p_1)}{(1)\ldots(n+1)\,.\,(1)\ldots(p_1+1)}\cdot\frac{(1)\ldots(n+p_2)}{(1)\ldots(n)\,.\,(1)\ldots(p_2)}$$

$$\times\begin{vmatrix} (p_1+1) & (p_2) \\ x\,(n) & (n+1) \end{vmatrix};$$

$$GF(p_1p_2p_3;\, n)$$

$$= \frac{(1)\ldots(n+p_1)}{(1)\ldots(n+2)\,.\,(1)\ldots(p_1+2)}\cdot\frac{(1)\ldots(n+p_2)}{(1)\ldots(n+1)\,.\,(1)\ldots(p_2+1)}\cdot\frac{(1)\ldots(n+p_3)}{(1)\ldots(n)\,.\,(1)\ldots(p_3)}$$

$$\times\begin{vmatrix} (p_1+1)\,(p_1+2) & (p_2)\,(p_2+1) & x\,(p_3-1)\,(p_3) \\ x\,(n)\,(p_1+2) & (n+1)\,(p_2+1) & (n+2)\,(p_3) \\ x^3\,(n-1)\,(n) & x\,(n)\,(n+1) & (n+1)\,(n+2) \end{vmatrix};$$

$$GF(l;\, 3;\, n) = \frac{(n+3)\ldots(n+l+2)}{(3)\ldots(l+2)}\cdot\frac{(n+2)\ldots(n+l+1)}{(2)\ldots(l+1)}\cdot\frac{(n+1)\ldots(n+l)}{(1)\ldots(l)};$$

$$GF(l;\, m;\, n)$$

$$= \frac{(n+m)\ldots(n+l+m-1)}{(m)\ldots(l+m-1)}\cdot\frac{(n+m-1)\ldots(n+l+m-2)}{(m-1)\ldots(l+m-2)}\cdots\cdots\frac{(n+1)\ldots(n+l)}{(1)\ldots(l)};$$

$$GF(l;\, m;\, \infty) = \frac{1}{(m)\ldots(l+m-1)\,.\,(m-1)\ldots(l+m-2)\,\ldots\ldots\,(1)\ldots(l)};$$

$$GF(l;\, \infty;\, \infty) = \frac{1}{(1)\,(2)\ldots(l)\,.\,(2)\,(3)\ldots(l+1)\,.\,(3)\,(4)\ldots(l+2)\,\ldots\ldots\,\text{ad inf.}};$$

$$GF(\infty;\, \infty;\, \infty) = \frac{1}{(1)\,(2)^2\,(3)^3\,(4)^4\ldots\ldots\,\text{ad inf.}}.$$

496. The expression of $GF(p_1p_2\ldots p_m;\, n)$ is not altered by substituting for $p_1p_2\ldots p_m$ the partition conjugate to it; though the *form* is changed. Interesting identities therefore present themselves.

In general, the solid graphs with which $GF(p_1p_2\ldots p_m;\, n)$ deals are those *contained* in the graph

$$\begin{array}{l|llllll}
O & & & & & & x \\
\hline
 & n & n & n & n & n\ \ldots\ldots & p_1 \text{ nodes in breadth} \\
 & n & n & n & n & \ldots\ldots & p_2 \quad\quad ,, \\
 & n & n & n & n & \ldots\ldots & p_3 \quad\quad ,, \\
 & n & n & & & \ldots\ldots & p_4 \quad\quad ,, \\
 & n & n & & & & \\
 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
y & n & n & & & \ldots\ldots & p_m \quad\quad ,,
\end{array}$$

where n denotes a pile of n nodes in direction of the axis of z.

This graph has six aspects which may be exhibited in the following manner, so as to put in evidence the numbers l, m, n.

Instead of the partition $p_1 p_2 \ldots p_m$ adopt the notation

$$l_1 l_2 \ldots l_m,$$

and let $m_1 m_2 \ldots m_l$ be the partition conjugate to it, so that

$$l_1 = l, \quad m_1 = m.$$

If we rotate the above graph about the axis of y through a right angle we get

$$\begin{array}{|llllll}
\hline
l_1 & l_1 & l_1 & l_1 & l_1 & \ldots\ldots n \text{ nodes in breadth} \\
l_2 & l_2 & l_2 & l_2 & l_2 & \ldots\ldots \quad\quad ,, \\
l_3 & l_3 & l_3 & l_3 & l_3 & \ldots\ldots \quad\quad ,, \\
l_4 & l_4 & l_4 & l_4 & l_4 & \ldots\ldots \quad\quad ,, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots \\
l_m & l_m & l_m & l_m & l_m & \ldots\ldots \quad\quad ,,
\end{array}$$

which is a graph, uniformly n nodes in breadth, parallel to the axis of x, and standing on a plane graph, m nodes long, parallel to the axis of y; the sth row being uniformly of height l_s, parallel to the axis of z. Rotating to obtain the other aspects we find that

$$GF(l_1 l_2 \ldots l_m; n)$$

enumerates plane partitions specified by

(i) rows $l_1, l_2, \ldots l_m$ in length respectively with a limit of part magnitude n;

(ii) rows $m_1, m_2, \ldots m_l$ in length respectively with a limit of part magnitude n;

(iii) n columns and m rows, the part magnitudes in the rows being limited by the numbers $l_1, l_2, \ldots l_m$ respectively;

(iv) n columns and l rows, the part magnitudes in the rows being limited by the numbers $m_1, m_2, \ldots m_l$ respectively;

(v) m columns and n rows, the part magnitudes in the columns being limited by the numbers $l_1, l_2, \ldots l_m$ respectively;

(vi) l columns and n rows, the part magnitudes in the columns being limited by the numbers $m_1, m_2, \ldots m_l$ respectively.

497. Valuable information concerning line or one-dimensional partitions is furnished by putting $n = 1$ in the general formula.

The partitions which are then enumerated are those in which every part is unity, there being *not more than* p_r units in the rth row:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & & \\ 1 & 1 & & & \\ 1 & & & & \end{matrix}$$

In fact each partition is derived from the graph of a unipartite line partition by substituting units for nodes, and the number of units in each row is limited not to exceed a certain number appertaining to the row.

If we add up the units in each row we obtain a line partition

$$(p_1'p_2'p_3' \dots p_m'),$$

in which

$$p_s' \not> p_s,$$

$p_1, p_2, \dots p_m$ being given numbers in descending order of magnitude. The partition $(p_1'p_2' \dots p_m')$ may be said to be "contained" in the partition $(p_1p_2 \dots p_m)$, and the formula shews how many partitions of given content are contained in the given partition $(p_1p_2 \dots p_m)$.

Putting $n = 1$ in the expression of $GF(p_1p_2p_3;\ n)$ we find after a slight reduction

$$GF(p_1p_2p_3;\ 1) = \frac{(\mathbf{p}_3+1)}{(1)(2)(3)} \begin{vmatrix} (\mathbf{p}_1+1) & (\mathbf{p}_2)(\mathbf{p}_2+1) & x(\mathbf{p}_3-1)(\mathbf{p}_3) \\ x & \frac{(2)}{(1)}(\mathbf{p}_2+1) & \frac{(3)}{(1)}(\mathbf{p}_3) \\ 0 & x & \frac{(2)(3)}{(1)(2)} \end{vmatrix},$$

and denoting

$$\frac{(1)(2)\dots(\mathbf{p})}{(1)(2)\dots(\mathbf{q})\,.\,(1)(2)\dots(\mathbf{p}-\mathbf{q})} \text{ by } X_{pq} \text{ or } X_{p,p-q},$$

we may write

$$GF(p_1p_2;\ 1) = \frac{(\mathbf{p}_2+1)}{(1)(2)} \begin{vmatrix} (\mathbf{p}_1+1) & (\mathbf{p}_2) \\ x & X_{21} \end{vmatrix},$$

$$GF(p_1p_2p_3;\ 1) = \frac{(\mathbf{p}_3+1)}{(1)(2)(3)} \begin{vmatrix} (\mathbf{p}_1+1) & (\mathbf{p}_2)(\mathbf{p}_2+1) & x(\mathbf{p}_3-1)(\mathbf{p}_3) \\ x & X_{21}(\mathbf{p}_2+1) & X_{31}(\mathbf{p}_3) \\ 0 & x & X_{32} \end{vmatrix},$$

$$GF(p_1p_2p_3p_4;\ 1)$$

$$= \frac{1}{(1)(2)(3)(4)} \begin{vmatrix} (\mathbf{p}_1+1) & (\mathbf{p}_2)(\mathbf{p}_2+1) & x(\mathbf{p}_3-1)(\mathbf{p}_3)(\mathbf{p}_3+1) & x^3(\mathbf{p}_4-2)(\mathbf{p}_4-1)(\mathbf{p}_4)(\mathbf{p}_4+1) \\ x & X_{21}(\mathbf{p}_2+1) & X_{31}(\mathbf{p}_3)(\mathbf{p}_3+1) & xX_{41}(\mathbf{p}_4-1)(\mathbf{p}_4)(\mathbf{p}_4+1) \\ 0 & x & X_{32}(\mathbf{p}_3+1) & X_{42}(\mathbf{p}_4)(\mathbf{p}_4+1) \\ 0 & 0 & x & X_{43}(\mathbf{p}_4+1) \end{vmatrix}$$

In this formula of the order four the factor (p_4+1) has been entere in the last column of the determinant so that the law may be the mo apparent to the reader. x appears in each constituent in the minor diagon which lies below the principal diagonal. In the minor diagonals which l above the principal one the powers of x are x^0, x, $x^{\binom{3}{2}}$, $x^{\binom{4}{2}}$,... etc., th figurate numbers of the third order appearing as exponents.

Ex. gr. $$GF(p_1p_2;\ 1)=\frac{1-x^{p_2+1}}{(1-x)(1-x^2)}(1+x^{p_2+1}-x^{p_1+1}-x^{p_1+2}),$$

and we find particularly that

$$GF(42;\ 1)=1+x+2x^2+2x^3+3x^4+2x^5+x^6,$$

the partitions enumerated by the coefficients being

zero; 1; 11, 2; 21, 3; 22, 31, 4; 32, 41; 42,

viz. all those that are contained in 42.

498. To enumerate the partitions contained in $(p_1p_2\dots p_m)$ is the sa as to find the sum of the coefficients in the expanded expression of

$$GF(p_1p_2\dots p_m;\ 1).$$

We have therefore to put $x=1$ in the above formula. This is done putting every bracketed number (s) equal to s and $X_{pq}=\binom{p}{q}$.

Thus for the order two we find the number $\frac{1}{2}(p_2+1)(2p_1-p_2+2)$.

These results are unchanged when for $(p_1p_2\dots p_m)$ is substituted t conjugate partition.

CHAPTER III

PARTITIONS *IN SOLIDO*

499. In this chapter we make a preliminary investigation concerning solid partitions where the parts are placed at the nodes of a solid graph and descending order of magnitude is insisted upon in the directions of the three axes of the graph.

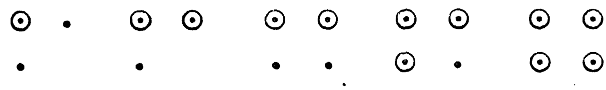
The remark may be made that if a unit be placed at each node of a solid graph and the graph be projected upon either of the coordinate planes by addition of units we thus obtain a plane partition in the plane of projection. In fact plane partitions are in theory coincident with solid partitions in which the part magnitude is restricted not to exceed unity. It will be remembered that similarly it was found that plane partitions in which the part magnitude is restricted not to exceed unity are coincident in theory with line partitions.

The enumeration of solid partitions is also connected with permutations derived from a solid lattice and with lattice functions. If we consider the three-dimensional system of nodes we regard it as a system of layers of nodes parallel to the xy-plane, each layer being the graph of a unipartite partition. If a particular layer, the sth from the xy-plane, be the graph of the partition $(s_1 s_2 \dots s_m)$ we associate with it the assemblage

$$\alpha_s^{s_1} \beta_s^{s_2} \dots \mu_s^{s_m}.$$

Thus taking the whole of the layers we get an assemblage of Greek letters involving as many suffixes as layers. Of such an assemblage we form lattice permutations. These are such that (i) looking merely at a particular suffix s the permutation must be a lattice one, (ii) looking merely at a particular Greek letter, say θ, the permutation must be a lattice one. In forming the lattice functions from these permutations we note (i) a letter which comes immediately before another which is prior to it in alphabetical order, (ii) a suffix which comes immediately before a suffix prior to it in numerical order provided that both suffixes are attached to the same Greek letter.

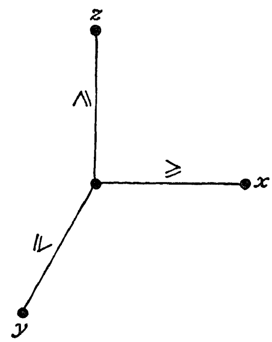
500. If we restrict attention to the cases in which not more than t
nodes appear in the direction of either axis the distinct solid graphs are



otherwise denoted by plane partitions

21	22	22	22	22
1	1	11	21	22

Taking the first of these we have four nodes, shewn in the diagra in perspective, at which numbers are to be placed so as to obey t inequalities indicated. The two layers parallel to the xy-plane have t specification 21 and 1 read parallel to the x-axis. We associate with t



solid graph the assemblages $\alpha_1\alpha_1\beta_1$, α_2 for the layers respectively and co sider the lattice permutations of the assemblage $\alpha_1\alpha_1\beta_1\alpha_2$. The lattice l must be obeyed by each of the triads $\alpha_1\alpha_1\beta_1$, $\alpha_1\alpha_1\alpha_2$. We find that t permutations are six in number, viz.:

	Index
$\alpha_1\alpha_1\beta_1 \mid \alpha_2$	3
$\alpha_1\beta_1 \mid \alpha_1\alpha_2$	2
$\alpha_1\alpha_1\alpha_2\beta_1$	0
$\alpha_1\beta_1 \mid \alpha_2 \mid \alpha_1$	5
$\alpha_1\alpha_2 \mid \alpha_1\beta_1$	2
$\alpha_1\alpha_2\beta_1 \mid \alpha_1$	3

leading to the lattice function

$$1 + (2x^2 + 2x^3) + x^5,$$

the sub-lattice functions being indicated.

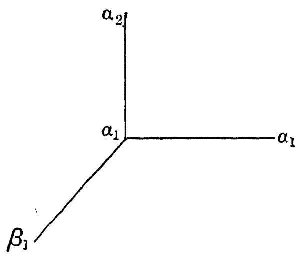
501. Denote by $$GF(21;\ 1;\ n)$$
the function which enumerates the partitions when the part magnitude is restricted by the number n. Then the principles established in preceding chapters give the results:

$$GF(21;\ 1;\ \infty)=\frac{1+(2x^2+2x^3)+x^5}{(1)(2)(3)(4)},$$

$$GF(21;\ 1;\ n)$$
$$=\frac{(n+1)(n+2)(n+3)(n+4)+(2x^2+2x^3)(n)(n+1)(n+2)(n+3)+x^5(n-1)(n)(n+1)(n+2)}{(1)(2)(3)(4)},$$

$$\Sigma g^n\, GF(21;\ 1;\ n)=\frac{1+g(2x^2+2x^3)+g^2x^5}{(1-g)(1-gx)(1-gx^2)(1-gx^3)(1-gx^4)}.$$

502. In fact the assemblage of Greek letters must be regarded as connected with the solid lattice in the manner shewn in the figure, and the property of a lattice permutation of the assemblage is that if a line be drawn between any two letters the nodes which correspond to the letters to the left of the line constitute a complete or incomplete lattice. Thus the permutation cannot commence with β_1 or α_2 because neither of the nodes corresponding thereto constitutes a lattice. The letter α_1 must be at the origin of the lattice. In the present instance every permutation which commences with α_1 forms progressively a system of nodes which always constitutes a lattice. It is clear that, fixing the attention upon the nodes which lie in any plane parallel to one of the coordinate planes, the corresponding letters must occur in a lattice permutation of those letters. If the nodes lie in ξ planes parallel to the yz-plane, η planes parallel to the zx-plane, and ζ planes parallel to the xy-plane, the whole assemblage of letters must satisfy $\xi+\eta+\zeta$ conditions; for that number of collections of letters drawn from the assemblage must be in a lattice permutation. In the above simple case



$$\xi=2,\quad \eta=2,\quad \zeta=2,$$

but the six conditions practically reduce to one only, viz. that associated with either of the planes xy, zx. It is easy to see that in any solid lattice each layer of nodes parallel to a coordinate plane must be a plane lattice and that if, in proceeding from left to right of the permutation of the assemblage, at any moment any such plane lattice be deficient, then the solid lattice will be deficient and cannot be in correspondence with a lattice permutation of the assemblage. The full number of conditions for a lattice permutation, viz. $\xi+\eta+\zeta$, will most frequently suffer some reduction, as those conditions

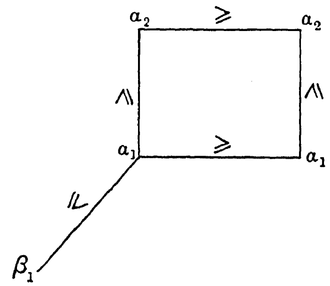
may be either trivial or not independent. In the above case the plane which contain the single nodes α_1, β_1, α_2 give three trivial conditions and th planes yz, xy conditions implied by that given by the plane zx.

The lattice permutations are then connected, as in the case of plan partitions, with non-overlapping Diophantine inequalities and the invest gation proceeds with the results given.

503. Passing now to the solid graph

$$\begin{array}{cc} \odot & \odot \\ \bullet & \end{array} \equiv \begin{array}{c} 22 \\ 1 \end{array}$$

we consider the system



and in forming the lattice permutations of the assemblage $\alpha_1\alpha_1\alpha_2\alpha_2\beta_1$ we hav only to take care that the letters in the plane zx are in lattice permutation There are eight lattice permutations:

	Index		Index
$\alpha_1\alpha_1\alpha_2\alpha_2\beta_1$	0	$\alpha_1\alpha_1\beta_1\alpha_2\alpha_2$	3
$\alpha_1\alpha_2\alpha_1\alpha_2\beta_1$	2	$\alpha_1\alpha_2\beta_1\alpha_1\alpha_2$	3
$\alpha_1\alpha_1\alpha_2\beta_1\alpha_2$	4	$\alpha_1\beta_1\alpha_1\alpha_2\alpha_2$	2
$\alpha_1\alpha_2\alpha_1\beta_1\alpha_2$	6	$\alpha_1\beta_1\alpha_2\alpha_1\alpha_2$	5

and the lattice function is

$$1 + (2x^2 + 2x^3 + x^4) + x^5 + x^6,$$

yielding the results

$$GF(21;\ 2;\ \infty) = \frac{1 + (2x^2 + 2x^3 + x^4) + (x^5 + x^6)}{(1)(2)(3)(4)(5)},$$

$$GF(21;\ 2;\ n)$$

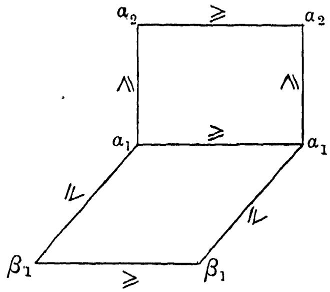
$$= \frac{(n+1)\ldots(n+5) + (2x^2+2x^3+x^4)(n)\ldots(n+4) + (x^5+x^6)(n-1)\ldots(n+3)}{(1)(2)\ldots(5)},$$

$$\Sigma g^n\, GF(21;\ 2;\ n) = \frac{1 + g(2x^2 + 2x^3 + x^4) + g^2(x^5 + x^6)}{(1-g)(1-gx)(1-gx^2)(1-gx^3)(1-gx^4)(1-gx^5)}.$$

504. Next we come to the solid lattice

$$\begin{matrix}\odot & \odot \\ \bullet & \bullet\end{matrix} \equiv \begin{matrix}22\\11\end{matrix}$$

and consider the system



The lattice permutations of the assemblage $\alpha_1\alpha_1\alpha_2\alpha_2\beta_1\beta_1$ must involve the collections $\alpha_1\alpha_1\beta_1\beta_1$, $\alpha_1\alpha_1\alpha_2\alpha_2$ in lattice permutations and we do not attend to other conditions. We have sixteen permutations:

	Index		Index
$\alpha_1\alpha_1\beta_1\beta_1\alpha_2\alpha_2$	4	$\alpha_1\alpha_1\beta_1\alpha_2\alpha_2\beta_1$	3
$\alpha_1\beta_1\alpha_1\beta_1\alpha_2\alpha_2$	6	$\alpha_1\beta_1\alpha_1\alpha_2\alpha_2\beta_1$	2
$\alpha_1\alpha_1\beta_1\alpha_2\beta_1\alpha_2$	8	$\alpha_1\alpha_1\alpha_2\beta_1\alpha_2\beta_1$	4
$\alpha_1\beta_1\alpha_1\alpha_2\beta_1\alpha_2$	7	$\alpha_1\beta_1\alpha_2\alpha_1\alpha_2\beta_1$	5
$\alpha_1\alpha_1\alpha_2\beta_1\beta_1\alpha_2$	5	$\alpha_1\alpha_2\alpha_1\beta_1\alpha_2\beta_1$	6
$\alpha_1\beta_1\alpha_2\alpha_1\beta_1\alpha_2$	10	$\alpha_1\alpha_2\beta_1\alpha_1\alpha_2\beta_1$	3
$\alpha_1\alpha_2\alpha_1\beta_1\beta_1\alpha_2$	7	$\alpha_1\alpha_1\alpha_2\alpha_2\beta_1\beta_1$	0
$\alpha_1\alpha_2\beta_1\alpha_1\beta_1\alpha_2$	8	$\alpha_1\alpha_2\alpha_1\alpha_2\beta_1\beta_1$	2

and the lattice function

$$1 + (2x^2 + 2x^3 + 2x^4 + x^5) + (x^5 + 2x^6 + 2x^7 + 2x^8) + x^{10}.$$

When we come to examine these permutations we find that those which terminate with β_1 give rise to a portion of the lattice function which is equal to the whole lattice function derived from the permutations of the assemblage $\alpha_1\alpha_1\alpha_2\alpha_2\beta_1$, and necessarily so because no component of the index can arise by the contact of any letter with the terminating β_1. Also as regards those which terminate with α_2 the permutations are of two kinds. Those lattice permutations of the collection $\alpha_1\alpha_1\beta_1\beta_1\alpha_2$ which terminate with α_2 pass on the same index when another α_2 is added, but those which terminate with β_1 add the number 5 to it when α_2 is added.

The reader will now have no difficulty in realising the relation which follows from these observations, viz.:

$$L(22;\ 2;\ \infty) = L(21;\ 2;\ \infty) + x^5L(21;\ 1;\ \infty) + x^4L(22;\ \infty),$$

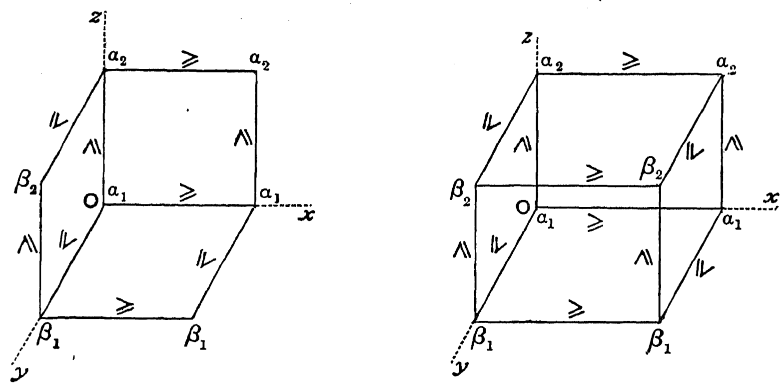
because those permutations, written out above, which terminate with $\beta_1\alpha_2$, are the whole of the lattice permutations of the assemblage $\alpha_1\alpha_1\beta_1\alpha_2$; and those which terminate with $\alpha_2\alpha_2$ are the whole of the lattice permutations of the assemblage $\alpha_1\alpha_1\beta_1\beta_1$, the factor x^4 appearing because of the conjunction of β_1 with the penultimate α_2.

The formulæ for $GF\,(22;\ 2;\ \infty)$, $GF\,(22;\ 2;\ n)$ and $\Sigma g^n GF\,(22;\ 2;\ n)$ are written down as in the other cases and need not be particularly given.

505. The remaining cases for consideration are the solid lattices

$$\begin{matrix}\odot & \odot \\ \odot & \bullet\end{matrix} \equiv \begin{matrix}22\\21\end{matrix}, \qquad \begin{matrix}\odot & \odot \\ \odot & \odot\end{matrix} \equiv \begin{matrix}22\\22\end{matrix},$$

and they are so intimately connected that it is convenient to consider them together.

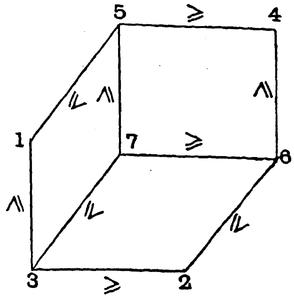


The assemblages of letters are $\alpha_1\alpha_1\beta_1\beta_1\alpha_2\alpha_2\beta_2$ and $\alpha_1\alpha_1\beta_1\beta_1\alpha_2\alpha_2\beta_2\beta_2$ respectively. If we write down any lattice permutation of the first of these we can obtain a lattice permutation of the second by the addition of β_2 as last letter, and since every lattice permutation of the second *must* end with β_2 it is clear that each assemblage has the same number of lattice permutations. Moreover the addition of β_2 to the lattice permutations of the first introduces no new component to the index. Hence the two solid lattices we are considering lead to the same lattice function and we have

$$L\,(22;\ 21;\ \infty) = L\,(22;\ 22;\ \infty).$$

506. There is a one-to-one correspondence for a solid lattice of w nodes between the arrangements of w different integers placed at the nodes so as to satisfy the inequalities marked between adjacent nodes and the allied lattice permutations. We will illustrate this in the case of the seven nodes, which form seven out of eight summits of a cube, shewn in the left-hand

diagram of the two above given. We will at the same time determine the lattice function. One arrangement of integers is



where it will be observed that the inequalities are satisfied.

We write down the corresponding lattice permutation by writing underneath the numbers, which are arranged in descending order, the corresponding suffixed Greek letters as depicted in the first seven-node diagram. Thus in the present example

$$\begin{array}{ccccccc}7&6&5&4&3&2&1\\ \alpha_1&\alpha_1&\alpha_2&\alpha_2&\beta_1&\beta_1&\beta_2\end{array}$$

and a lattice permutation of zero index is derived. In the following scheme the lower layer and upper layer of numbers are given on the left and on the right respectively:

$$\begin{array}{cccc}7&6&5&4\\3&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&4&6&2\\5&3&1&\end{array}$$
$$\alpha_1\,\alpha_1\,\alpha_2\,\alpha_2\,\beta_1\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\alpha_2\,\beta_1|\alpha_1\,\beta_1|\alpha_2\,\beta_2$$

$$\begin{array}{cccc}7&5&6&4\\3&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&4&6&2\\5&1&3&\end{array}$$
$$\alpha_1\,\alpha_2|\alpha_1\,\alpha_2\,\beta_1\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\alpha_2\,\beta_1|\alpha_1\,\beta_2|\alpha_2\,\beta_1$$

$$\begin{array}{cccc}7&5&4&3\\6&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&5&4&3\\6&1&2&\end{array}$$
$$\alpha_1\,\beta_1|\alpha_1\,\alpha_2\,\alpha_2\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\beta_1|\alpha_1\,\alpha_2\,\alpha_2\,\beta_2|\beta_1$$

$$\begin{array}{cccc}7&6&4&3\\5&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&5&4&1\\6&3&2&\end{array}$$
$$\alpha_1\,\alpha_1\,\beta_1|\alpha_2\,\alpha_2\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\beta_1|\alpha_1\,\alpha_2\,\beta_1\,\beta_2|\alpha_2$$

$$\begin{array}{cccc}7&4&6&3\\5&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&6&4&3\\5&1&2&\end{array}$$
$$\alpha_1\,\alpha_2\,\beta_1|\alpha_1\,\alpha_2\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\alpha_1\,\beta_1|\alpha_2\,\alpha_2\,\beta_2|\beta_1$$

$$\begin{array}{cccc}7&6&5&3\\4&2&1&\end{array}\qquad\qquad\begin{array}{cccc}7&6&4&1\\5&3&2&\end{array}$$
$$\alpha_1\,\alpha_1\,\alpha_2\,\beta_1|\alpha_2\,\beta_1\,\beta_2\qquad\qquad\alpha_1\,\alpha_1\,\beta_1|\alpha_2\,\beta_1\,\beta_2|\alpha_2$$

$\begin{array}{cccc}7&6&3&2\\5&4&1&\end{array}$ $\alpha_1\,\alpha_1\,\beta_1\,\beta_1\vert\alpha_2\,\alpha_2\,\beta_2$	$\begin{array}{cccc}7&4&6&3\\5&1&2&\end{array}$ $\alpha_1\,\alpha_2\,\beta_1\vert\alpha_1\,\alpha_2\,\beta_2\vert\beta_1$
$\begin{array}{cccc}7&3&6&2\\5&1&4&\end{array}$ $\alpha_1\,\alpha_2\,\beta_1\,\beta_2\vert\alpha_1\,\alpha_2\,\beta_1$	$\begin{array}{cccc}7&4&6&1\\5&3&2&\end{array}$ $\alpha_1\,\alpha_2\,\beta_1\vert\alpha_1\,\beta_1\,\beta_2\vert\alpha_2$
$\begin{array}{cccc}7&6&5&2\\4&3&1&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\beta_1\,\beta_1\vert\alpha_2\,\beta_2$	$\begin{array}{cccc}7&6&5&3\\4&1&2&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\beta_1\vert\alpha_2\,\beta_2\vert\beta_1$
$\begin{array}{cccc}7&6&5&2\\4&1&3&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\beta_1\,\beta_2\vert\alpha_2\,\beta_1$	$\begin{array}{cccc}7&6&3&1\\5&4&2&\end{array}$ $\alpha_1\,\alpha_1\,\beta_1\,\beta_1\vert\alpha_2\,\beta_2\vert\alpha_2$
$\begin{array}{cccc}7&6&5&4\\3&1&2&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\alpha_2\,\beta_1\,\beta_2\vert\beta_1$	$\begin{array}{cccc}7&3&6&1\\5&2&4&\end{array}$ $\alpha_1\,\alpha_2\,\beta_1\,\beta_2\vert\alpha_1\,\beta_1\vert\alpha_2$
$\begin{array}{cccc}7&6&5&1\\4&3&2&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\beta_1\,\beta_1\,\beta_2\vert\alpha_2$	$\begin{array}{cccc}7&6&5&1\\4&2&3&\end{array}$ $\alpha_1\,\alpha_1\,\alpha_2\,\beta_1\,\beta_2\vert\beta_1\vert\alpha_2$
$\begin{array}{cccc}7&4&5&3\\6&2&1&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\vert\alpha_1\,\alpha_2\,\beta_1\,\beta_2$	$\begin{array}{cccc}7&4&5&2\\6&3&1&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\vert\alpha_1\,\beta_1\vert\alpha_2\,\beta_2$
$\begin{array}{cccc}7&5&6&3\\4&2&1&\end{array}$ $\alpha_1\,\alpha_2\vert\alpha_1\,\beta_1\vert\alpha_2\,\beta_1\,\beta_2$	$\begin{array}{cccc}7&4&5&2\\6&1&3&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\vert\alpha_1\,\beta_2\vert\alpha_2\,\beta_1$
$\begin{array}{cccc}7&5&3&2\\6&4&1&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_1\,\beta_1\vert\alpha_2\,\alpha_2\,\beta_2$	$\begin{array}{cccc}7&4&5&3\\6&1&2&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\vert\alpha_1\,\alpha_2\,\beta_2\vert\beta_1$
$\begin{array}{cccc}7&3&5&2\\6&1&4&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\,\beta_2\vert\alpha_1\,\alpha_2\,\beta_1$	$\begin{array}{cccc}7&4&5&1\\6&3&2&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\vert\alpha_1\,\beta_1\,\beta_2\vert\alpha_2$
$\begin{array}{cccc}7&5&6&2\\4&3&1&\end{array}$ $\alpha_1\,\alpha_2\vert\alpha_1\,\beta_1\,\beta_1\vert\alpha_2\,\beta_2$	$\begin{array}{cccc}7&5&6&3\\4&1&2&\end{array}$ $\alpha_1\,\alpha_2\vert\alpha_1\,\beta_1\vert\alpha_2\,\beta_2\vert\beta_1$
$\begin{array}{cccc}7&5&6&2\\4&1&3&\end{array}$ $\alpha_1\,\alpha_2\vert\alpha_1\,\beta_1\,\beta_2\vert\alpha_2\,\beta_1$	$\begin{array}{cccc}7&5&3&1\\6&4&2&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_1\,\beta_1\vert\alpha_2\,\beta_2\vert\alpha_2$
$\begin{array}{cccc}7&5&4&2\\6&3&1&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_1\,\alpha_2\,\beta_1\vert\alpha_2\,\beta_2$	$\begin{array}{cccc}7&3&5&1\\6&2&4&\end{array}$ $\alpha_1\,\beta_1\vert\alpha_2\,\beta_2\vert\alpha_1\,\beta_1\vert\alpha_2$

$$\begin{array}{llll} 7 & 5 & 4 & 2 \\ 6 & 1 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\beta_1|\alpha_1\,\alpha_2\,\beta_2|\alpha_2\,\beta_1} \end{array} \qquad \begin{array}{llll} 7 & 5 & 6 & 1 \\ 4 & 2 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_2|\alpha_1\,\beta_1\,\beta_2|\beta_1|\alpha_2} \end{array}$$

$$\begin{array}{llll} 7 & 6 & 4 & 2 \\ 5 & 3 & 1 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_1\,\beta_1|\alpha_2\,\beta_1|\alpha_2\,\beta_2} \end{array} \qquad \begin{array}{llll} 7 & 5 & 4 & 1 \\ 6 & 2 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\beta_1|\alpha_1\,\alpha_2\,\beta_2|\beta_1|\alpha_2} \end{array}$$

$$\begin{array}{llll} 7 & 6 & 4 & 2 \\ 5 & 1 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_1\,\beta_1|\alpha_2\,\beta_2|\alpha_2\,\beta_1} \end{array} \qquad \begin{array}{llll} 7 & 6 & 4 & 1 \\ 5 & 2 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_1\,\beta_1\,\alpha_2\,\beta_2|\beta_1|\alpha_2} \end{array}$$

$$\begin{array}{llll} 7 & 5 & 6 & 4 \\ 3 & 1 & 2 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_2|\alpha_1\,\alpha_2\,\beta_1\,\beta_2|\beta_1} \end{array} \qquad \begin{array}{llll} 7 & 4 & 6 & 1 \\ 5 & 2 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_2\,\beta_1|\alpha_1\,\beta_2|\beta_1|\alpha_2} \end{array}$$

$$\begin{array}{llll} 7 & 5 & 6 & 1 \\ 4 & 3 & 2 & \\ \multicolumn{4}{l}{\alpha_1\,\alpha_2|\alpha_1\,\beta_1\,\beta_1\,\beta_2|\alpha_2} \end{array} \qquad \begin{array}{llll} 7 & 4 & 5 & 1 \\ 6 & 2 & 3 & \\ \multicolumn{4}{l}{\alpha_1\,\beta_1|\alpha_2|\alpha_1\,\beta_2|\beta_1|\alpha_2} \end{array}$$

507. There are 48 permutations. The dividing lines shew the order of the associated lattice function. Forming the lattice function in the ordinary way we find that

$$\begin{aligned} & L(22;\ 22;\ \infty) \\ &= 1 + (2x^2 + 2x^3 + 3x^4 + 2x^5 + 2x^6) \\ &+ (x^5 + 3x^6 + 4x^7 + 8x^8 + 4x^9 + 3x^{10} + x^{11}) \\ &+ (2x^{10} + 2x^{11} + 3x^{12} + 2x^{13} + 2x^{14}) + x^{16} \\ &= L(22;\ 21;\ \infty); \end{aligned}$$

and thence writing

$$L(22;22;\infty) = L_0 + L_1 + L_2 + L_3 + L_4,$$

we obtain

$$GF(22;22;\infty) = \frac{L(22;22;\infty)}{(1)(2)\dots(8)};$$

$$GF(22;21;\infty) = \frac{L(22;21;\infty)}{(1)(2)\dots(7)};$$

$$GF(22;22;n) = \frac{\Sigma L_t\,(n-t+1)(n-t+2)\dots(n-t+8)}{(1)(2)\dots(8)};$$

$$GF(22;21;n) = \frac{\Sigma L_t\,(n-t+1)(n-t+2)\dots(n-t+8)}{(1)(2)\dots(7)};$$

$$\Sigma g^n\,GF(22;22;n) = \frac{L_0 + gL_1 + g^2L_2 + g^3L_3 + g^4L_4}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^8)};$$

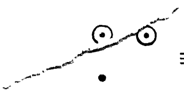
$$\Sigma g^n\,GF(22;21;n) = \frac{L_0 + gL_1 + g^2L_2 + g^3L_3 + g^4L_4}{(1-g)(1-gx)(1-gx^2)\dots(1-gx^7)}.$$

508. This completes the results up to the point where the number of nodes along each axis does not exceed two. It suffices to establish that by the method pursued we have potentially the complete solution of the problem of Solid Partitions. The discovery of the algebraic form of the lattice function in the general case has not been made. It appears to be a complicated and difficult question. A short discussion, however, may be given.

When we were speaking of solid lattices it was shewn that such a lattice might be read in either one, two, three, or six different ways, according to the degree of symmetry possessed by the lattice. It follows that the specification of a lattice by layers of nodes may be in one, two, three, or six different forms. Thus the notation $GF(21\,;\,1\,;\,n)$ is unique; but the three forms

$$GF(22\,;\,1\,;\,n),\quad GF(21\,;\,2\,;\,n),\quad GF(21\,;\,11\,;\,n)$$

are equivalent because each is associated with the same graph



$$\equiv \begin{matrix} 22 \\ 1 \end{matrix},$$

which has three aspects.

In particular the equivalence

$$GF(p_1p_2\,;\,q_1q_2\,;\,n) \equiv GF(p_1q_1\,;\,p_2q_2\,;\,n)$$

should be noted.

We have already met with one or two instances of relations between lattice functions of the new kind, so that it is not surprising to find that lattice functions and generating functions are connected by functional equations. By the same reasoning as was employed in a previous chapter we establish equations of the type

$$\begin{aligned}
&(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}_1)\,GF(p_1p_2\,;\,q_1\,;\,\infty) \\
&= GF(p_1-1\,p_2\,;\,q_1\,;\,\infty) + GF(p_1\,p_2-1\,;\,q_1\,;\,\infty) + GF(p_1p_2\,;\,q_1-1\,;\,\infty) \\
&- GF(p_1-1\,p_2-1\,;\,q_1\,;\,\infty) - GF(p_1-1\,p_2\,;\,q_1-1\,;\,\infty) - GF(p_1p_2-1\,;\,q_1-1\,;\,\infty) \\
&+ GF(p_1-1\,p_2-1\,;\,q_1-1\,;\,\infty),
\end{aligned}$$

and transforming to lattice functions

$$\begin{aligned}
&L(p_1p_2\,;\,q_1\,;\,\infty) \\
&= L(p_1-1\,p_2\,;\,q_1\,;\,\infty) + L(p_1\,p_2-1\,;\,q_1\,;\,\infty) + L(p_1p_2\,;\,q_1-1\,;\,\infty) \\
&- (\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}_1-1)\,\{L(p_1-1\,p_2-1\,;\,q_1\,;\,\infty) + L(p_1-1\,p_2\,;\,q_1-1\,;\,\infty) \\
&\qquad + L(p_1\,p_2-1\,;\,q_1-1\,;\,\infty)\} \\
&+ (\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}_1-2)\,(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q}_1-1)\,L(p_1-1\,p_2-1\,;\,q_1-1\,;\,\infty),
\end{aligned}$$

with the usual modifications when there are equalities between the numbers p_1, p_2, q_1. As a particular case

$$L(l\,1;1;\infty)=L(l-1\,1;1;\infty)+2L(l\,1;\infty)-\mathbf{(l+1)}\{2L(l-1\,1;\infty)+L(l;\infty)\}$$
$$+\mathbf{(l)}\,\mathbf{(l+1)}\,L(l-1;\infty).$$

Since $$L(l\,1;\infty)=1+x^2\frac{\mathbf{(l-1)}}{\mathbf{(1)}};\quad L(l;\infty)=1,$$

we find that

$$L(l\,1;1;\infty)=L(l-1\,1;1;\infty)+x^2(1+x)\frac{\mathbf{(l+1)}}{\mathbf{(1)}}-2x^{l+2},$$

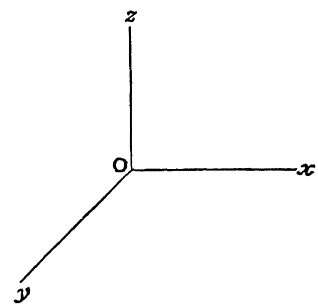
and now it is readily shewn that

$$\mathbf{(1)}^2\,L(l\,1;1;\infty)=\mathbf{(l+1)}\,\mathbf{(l+2)}-2x\,\mathbf{(1)}\,\mathbf{(l+2)}+x^2\,\mathbf{(1)}\,\mathbf{(2)}$$
$$=\begin{vmatrix}\mathbf{(l+2)} & \mathbf{(l+2)}-x\,\mathbf{(2)}\\ x\,\mathbf{(1)} & \mathbf{(l+1)}-x\,\mathbf{(1)}\end{vmatrix}.$$

CHAPTER IV

THE SYMMETRY APPERTAINING TO PARTITIONS

509. It has already been observed in Art. 428 that the three-dimensional graphs of plane partitions have either one, two, three, or six aspects, leading to one, two, three, or six different readings. There are thus four classes of graphs and partitions, and although their complete theory has not yet been discovered, the subject may be usefully discussed up to a certain point. Taking an origin and three coordinate axes, the graphs which permit of only one reading are symmetrical about the line through the origin equally inclined to the three coordinate planes (or axes). Upon that line, which may be called the diagonal of the solid graph, there must be one node, at the origin, and there may be any number. First consider the symmetrical graphs which have only the origin node upon the diagonal. Then the whole of the nodes must lie in one or other of the three coordinate planes. For every node upon the axis of x there must be corresponding



nodes upon the axes of y and z. For every node, not upon an axis, that lies in the plane xy there must be corresponding nodes in the planes yz and zx. The nodes that lie in any one of the three coordinate planes constitute a self-conjugate graph in two dimensions. We thus obtain three perfectly similar self-conjugate graphs, but they are not quite distinct, because there is overlapping due to the circumstance that the origin node lies in each of the three planes, while every node which lies upon one of the coordinate axes necessarily lies in each of two of the coordinate planes. We may therefore separate the nodes into three lots: (i) the origin node, (ii) nodes which lie upon the axes, the origin node excepted, (iii) the nodes which do not lie upon an axis. The nodes in (ii) occur in threes. The nodes in (iii) constitute in each coordinate plane the same self-conjugate graph, and therefore these nodes also occur in threes. The whole number of nodes must consequently be of the form $3m + 1$.

510. Let us now enumerate the graphs which have i nodes along each axis (the origin node included on each axis). We have then as a minimum $3i-2$ nodes. In the plane xy we may have any self-conjugate graph which involves exactly i nodes in the direction of each of the axes of x and y, *or*, if we omit those nodes which lie upon those axes in the plane xy, we have any self-conjugate graph which involves $i-1$ or fewer nodes in the direction of the axes of x and y. These graphs are enumerated by the function

$$(1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3}),$$

since we have seen that the nodes may be separated into angles of nodes, symmetrical about their origin, which involve numbers of nodes which are unrepeated uneven numbers. Clearly the external angle of nodes involves a maximum of $2i-3$ nodes.

Hence, since, as remarked, the nodes occur in threes, and there are $3i-2$ nodes upon the axes, we find that

$$x^{3i-2}(1+x^3)(1+x^9)(1+x^{15})\dots(1+x^{6i-9})$$

is the function which enumerates the solid graphs which have only one aspect, i nodes along each axis, and every node upon one of the coordinate planes. This leads us to the function

$$1+x+x^4(1+x^3)+x^7(1+x^3)(1+x^9)+\dots+x^{3i-2}(1+x^3)(1+x^9)\dots(1+x^{6i-9}),$$

to enumerate the graphs when there are i nodes or fewer upon each axis.

Also when $i=\infty$ to the function

$$1+x+x^4(1+x^3)+x^7(1+x^3)(1+x^9)+x^{10}(1+x^3)(1+x^9)(1+x^{15})+\dots \text{ ad inf.}$$

511. By viewing the matter in another manner we can obtain another form of the function which gives an interesting identity.

Suppose that in the plane xy we have four nodes along the diameter OA. This implies that the square OA is full of nodes, and, by symmetry, also each of the squares OB, OC must be full of nodes. (For diagram, see p. 260.)

Altogether we must have at least $4\times 3^2+1$ or 37 nodes, and $37=4^3-3^3$.

Without increasing the number of nodes upon the diameter OA, we may add nodes in the plane xy which constitute a plane graph made up of rows which do not exceed four nodes in length, the rows being taken parallel to the axis of y and terminating in the axis of x. But if we do this, symmetry demands that we add a similar graph in the plane xy, the rows being parallel to the axis of x and terminating in the axis of y; similar graphs also

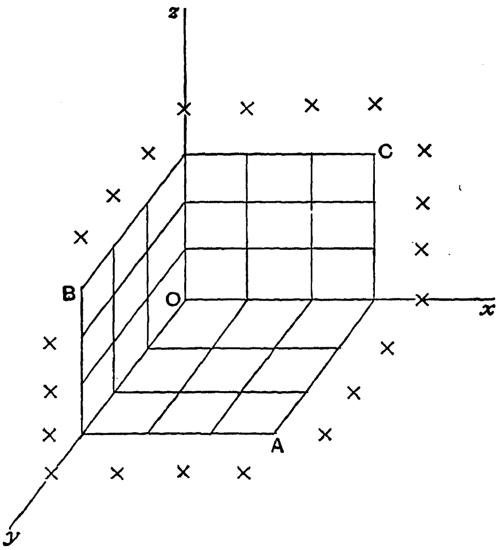
in the plane yz, with rows parallel to the axis of y, and terminating in the axis of z;

in the plane yz, with rows parallel to the axis of z and terminating in the axis of y;

in the plane zx, with rows parallel to the axis of z, and terminating the axis of x;

in the plane zx, with rows parallel to the axis of x, and terminating the axis of z.

These six graphs are not quite distinct; there is overlapping due to nodes that are added upon the axes. Thus a row of four nodes added in ea of the six half-planes adds not 6×4 or 24 nodes, but $24 - 3 = 21$ nodes. rows of three, two and one add $6 \times 3 - 3 = 15$, $6 \times 2 - 3 = 9$, and $6 \times 1 - 3 =$ nodes respectively.



The case of a row of four nodes is indicated by crosses in the diagram.

Hence our addition resolves itself into lots of 21 nodes, 15, 9, 3 no each any number of times repeated. We see then that if there are exac four nodes along the diameter OA we have an enumerating function

$$\frac{x^{4^3-3^3}}{(1-x^3)(1-x^9)(1-x^{15})(1-x^{21})},$$

and generally, if we have exactly $k-1$ nodes along the diameter OA, have an enumerating function

$$\frac{x^{(k-1)^3-(k-2)^3}}{(1-x^3)(1-x^9)(1-x^{15})\dots(1-x^{6k-9})}.$$

512. Thence we conclude that the graphs which have one reading o and every node in one of the coordinate planes, are enumerated by the seri

$$1+\frac{x}{1-x^3}+\frac{x^7}{(1-x^3)(1-x^9)}+\frac{x^{19}}{(1-x^3)(1-x^9)(1-x^{15})}+\dots$$
$$+\frac{x^{(k-1)^3-(k-2)^3}}{(1-x^3)\dots(1-x^{6k-9})}+\dots,$$

which, on expansion, must be identically the same as that previously found, viz.:

$$1 + x + x^4(1 + x^3) + x^7(1 + x^3)(1 + x^9) + \dots + x^{3i-2}(1 + x^3)\dots(1 + x^{6i-9}) + \dots.$$

We thus establish an interesting identity.

513. These partitions the nodes of whose graphs lie all upon the coordinate planes are represented by solid angles of nodes which are perfectly symmetrical in respect to the three coordinate planes. All other partitions which possess this xyz-symmetry are formed by fitting into one another two or more symmetrical solid angles of nodes so as to produce a regular solid graph. So far we have merely enumerated the external solid angles of nodes or boundary partitions. It is not difficult to construct a crude generating function arising from a number of Diophantine inequalities, but effectively the enumeration has not, up to the present time, been carried any further.

One further remark however may be made. From any one of these graphs we can immediately derive an infinite number of others of the same symmetrical nature; for if we take a square of nodes having i nodes in a side and form a symmetrical three-dimensional graph in the left-hand top corner, we may conceive the square

$$\begin{array}{cccccccc}
2 & 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

completed into a cube of i^3 nodes. The graph in the corner if now subtracted from the cube leaves a solid graph which if rotated through 180° presents a symmetrical graph. The two graphs are in fact supplementary the one of the other *quâ* a cube of i^3 nodes. Hence a symmetrical graph of $i^3 - w$ nodes in the general case where the graph of weight w has i or fewer nodes along each axis.

In the case above the graph of weight 4 gives rise to the graphs whose generating function is

$$x^4 + x^{3^3-4} + x^{4^3-4} + x^{5^3-4} + \dots + x^{i^3-4} + \dots.$$

Exceptionally in this case when $i = 2$, the graph itself is reproduced. Thus from the partition

$$\begin{array}{l} 321 \\ 21 \\ 1 \end{array}$$

we produce

332	4444	55555	
321	4443	55555	
21	4432	55554	etc. ad inf.
	4321	55543	
		55432	

514. Certain graphs possess symmetry in relation to the axes of x and y and do not possess symmetry in regard to the three coordinate planes. If we enumerate all the graphs which possess xy-symmetry we include those which possess xyz-symmetry. Those which possess xy-symmetry only have three aspects and give rise to plane partitions which read the same by rows as by columns. We therefore attempt the enumeration of plane partitions which have this property and understand that those which are perfectly symmetrical are included. Each layer of nodes parallel to the xy-plane is a self-conjugate graph. We may regard the whole graph as arising by superposing, when possible, such layers. This would be one method of attacking the problem, but it gives rise to Diophantine inequalities which are not easy to deal with. The method now given is preferable. Every plane partition possessing xy-symmetry naturally has successive rows of numbers of the same lengths as the successive columns of numbers. It is of the form

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \dots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \dots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \dots \\ \vdots & \vdots & \vdots \end{matrix}$$

where $\alpha_{st} = \alpha_{ts}$.

Consider the portion of the partition

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \dots \\ & \alpha_{22} & \alpha_{23} \dots \\ & & \alpha_{33} \dots \\ & & \ddots \end{matrix}$$

where the numbers are subject to the inequalities

$$\begin{matrix} \alpha_{11} \geqslant & \alpha_{12} \geqslant & \alpha_{13} \geqslant \dots \\ & \vee\!\!\!| & \vee\!\!\!| \\ & \alpha_{22} \geqslant & \alpha_{23} \geqslant \dots \\ & & \vee\!\!\!| \\ & & \alpha_{33} \geqslant \dots \\ & & \ddots \end{matrix}$$

and seek the sum

$$\begin{array}{l}\Sigma x_{11}^{\alpha_{11}}\ x_{12}^{\alpha_{12}}\ x_{13}^{\alpha_{13}}\ldots\\ \qquad\times x_{22}^{\alpha_{22}}\ x_{23}^{\alpha_{23}}\ldots\\ \qquad\qquad\times x_{33}^{\alpha_{33}}\ldots\\ \qquad\qquad\qquad\ddots\end{array}$$

for all systems of such numbers.

We are led to the crude function

$$\underset{\geqslant}{\Omega}\frac{1}{\begin{array}{l}(1-\lambda_{11}x_{11})\left(1-\dfrac{\lambda_{12}\mu_{11}}{\lambda_{11}}x_{12}\right)\left(1-\dfrac{\lambda_{13}\mu_{12}}{\lambda_{12}}x_{13}\right)\ldots\\ \qquad\left(1-\dfrac{\lambda_{22}}{\mu_{11}}x_{22}\right)\left(1-\dfrac{\lambda_{23}\mu_{22}}{\lambda_{22}\mu_{12}}x_{23}\right)\ldots\\ \qquad\qquad\left(1-\dfrac{\lambda_{33}}{\mu_{22}}x_{33}\right)\ldots\\ \qquad\qquad\qquad\ddots\end{array}},$$

wherein the auxiliary λ_{rc} refers to $\alpha_{rc}\geqslant\alpha_{r.c+1}$, and μ_{rc} refers to $\alpha_{r.c+1}\geqslant\alpha_{r+1.c+1}$.

Since in the enumeration in order to obtain the content or weight of the partition we have to double the parts which lie above the principal diagonal, to obtain an enumerating function we must put $x_{rr}=x$ for all values of r and $x_{rc}=x^2$ whenever $c>r$.

We thus find the function

$$\underset{\geqslant}{\Omega}\frac{1}{\begin{array}{l}(1-\lambda_{11}x)\left(1-\dfrac{\lambda_{12}\mu_{11}}{\lambda_{11}}x^2\right)\left(1-\dfrac{\lambda_{13}\mu_{12}}{\lambda_{12}}x^2\right)\ldots\\ \qquad\left(1-\dfrac{\lambda_{22}}{\mu_{11}}x\right)\left(1-\dfrac{\lambda_{23}\mu_{22}}{\lambda_{22}\mu_{12}}x^2\right)\ldots\\ \qquad\qquad\left(1-\dfrac{\lambda_{33}}{\mu_{22}}x\right)\ldots\\ \qquad\qquad\qquad\ddots\end{array}},$$

the crude expression of the sum

$$\Sigma x^{\alpha_{11}+\alpha_{22}+\alpha_{33}+\ldots+2(\alpha_{12}+\alpha_{13}+\ldots+\alpha_{23}+\ldots)}.$$

The coefficient of x^w in the sum is equal to the number of plane partitions of content w which possess xy-symmetry.

Suppose that the first row of the partition involves i numbers. We then have only to consider the first i columns of factors in the denominator of the crude function.

For $i=1$, we have $\dfrac{1}{1-x}$, which is clearly correct, for the partition consists for each weight of a single number (equal to the weight) at the origin of the graph or partition.

515. For $i=2$, we have

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_{11}x)\left(1-\frac{\mu_{11}}{\lambda_{11}}x^2\right)\left(1-\frac{1}{\mu_{11}}x\right)},$$

as the crude function which enumerates partitions of the form

$$\begin{matrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{matrix}$$

Eliminating μ_{11} it is

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_{11}x)\left(1-\frac{1}{\lambda_{11}}x^2\right)\left(1-\frac{1}{\lambda_{11}}x^3\right)},$$

and this is $$\frac{1}{(1-x)(1-x^3)(1-x^4)} \quad \text{or} \quad \frac{1}{(1)(3)(4)},$$

the correct enumerating function.

516. If we had required a real generating function we should have dealt with

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda_{11}x_{11})\left(1-\frac{\mu_{11}}{\lambda_{11}}x_{12}x_{21}\right)\left(1-\frac{1}{\mu_{11}}x_{22}\right)},$$

and have found $$\frac{1}{(1-x_{11})(1-x_{11}x_{12}x_{21})(1-x_{11}x_{12}x_{21}x_{22})},$$

shewing that the partitions can all be obtained from the partitions

$$\begin{matrix} 1 & \cdot \\ \cdot & \cdot \end{matrix}; \quad \begin{matrix} 1 & 1 \\ 1 & \cdot \end{matrix}; \quad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix},$$

by adding A times the first or $\begin{matrix} A & \cdot \\ \cdot & \cdot \end{matrix}$ to B times the second or $\begin{matrix} B & B \\ B & \cdot \end{matrix}$ and to C times the third or $\begin{matrix} C & C \\ C & C \end{matrix}$; thus producing the partition

$$\begin{matrix} A+B+C, & B+C, \\ B+C\ \ , & C\ \ , \end{matrix}$$

A, B, C being arbitrary positive integers. This is the general form of plane partition of the nature considered and there is no overlapping. There are no syzygies.

517. If we wish to restrict the part magnitude we proceed as on a former page, Art. 431, and write gx_{11} for x_{11}, at the same time introducing the additional denominator factor $1-g$. The generating function for a part

magnitude not exceeding n is then equal to the coefficient of g^n in the expanded function. We are thus led to the algebraic fraction

$$\frac{1}{(1-g)(1-gx)(1-gx^3)(1-gx^4)},$$

and herein the coefficient of g^n is readily found to be

$$\frac{(1-x^{n+1})(1-x^{n+3})(1-x^{2n+4})}{(1-x)(1-x^3)(1-x^4)}.$$

518. Proceeding to the case $i=3$, we find the crude function

$$\underset{\geqslant}{\Omega}\frac{1}{(1-\lambda_{11}x_{11})\left(1-\frac{\lambda_{12}\mu_{11}}{\lambda_{11}}x_{12}x_{21}\right)\left(1-\frac{\mu_{12}}{\lambda_{12}}x_{13}x_{31}\right)\left(1-\frac{\lambda_{22}}{\mu_{11}}x_{22}\right)\left(1-\frac{\mu_{22}}{\lambda_{22}\mu_{12}}x_{23}x_{32}\right)\left(1-\frac{1}{\mu_{22}}x_{33}\right)},$$

which successively exhibits the forms which follow, *in which for brevity and convenience* $x_{rc}x_{cr}$ *is written* x_{rcr}:

$$\underset{\geqslant}{\Omega}\frac{1}{(1-x_{11})(1-\lambda_{12}\mu_{11}x_{11}x_{121})\left(1-\frac{\mu_{12}}{\lambda_{12}}x_{131}\right)\left(1-\frac{\lambda_{22}}{\mu_{11}}x_{22}\right)\left(1-\frac{1}{\lambda_{22}\mu_{12}}x_{232}\right)\left(1-\frac{1}{\lambda_{22}\mu_{12}}x_{232}x_{33}\right)},$$

and thence

$$\underset{\geqslant}{\Omega}\frac{1}{(1-x_{11})(1-\lambda_{12}x_{11}x_{121})\left(1-\frac{1}{\lambda_{12}}x_{131}\right)(1-\lambda_{12}\lambda_{22}x_{11}x_{121}x_{22})\left(1-\frac{1}{\lambda_{12}\lambda_{22}}x_{131}x_{232}\right)\left(1-\frac{1}{\lambda_{12}\lambda_{22}}x_{131}x_{232}x_{33}\right)},$$

and thence

$$\underset{\geqslant}{\Omega}\frac{1}{(1-x_{11})(1-\lambda_{12}x_{11}x_{121})\left(1-\frac{1}{\lambda_{12}}x_{131}\right)(1-\lambda_{12}x_{11}x_{121}x_{22})(1-x_{11}x_{121}x_{22}x_{131}x_{232})(1-x_{11}x_{121}x_{22}x_{131}x_{232}x_{33})},$$

and finally by the formula given in Art. 348,

$$\frac{1-x_{11}^2x_{121}^2x_{22}x_{131}}{\left\{\begin{matrix}(1-x_{11})(1-x_{11}x_{121})(1-x_{11}x_{121}x_{22})(1-x_{11}x_{121}x_{131})(1-x_{11}x_{121}x_{22}x_{131})\\ \times(1-x_{11}x_{121}x_{22}x_{131}x_{232})(1-x_{11}x_{121}x_{22}x_{131}x_{232}x_{33})\end{matrix}\right\}}$$

This result shews that the fundamental products are

$$A_1 = x_{11}, \quad B_3 = x_{11}x_{12}x_{21}, \quad C_4 = x_{11}x_{12}x_{21}x_{22}, \quad D_5 = x_{11}x_{12}x_{21}x_{13}x_{31},$$

$$E_6 = x_{11}x_{12}x_{21}x_{22}x_{13}x_{31}, \quad F_8 = x_{11}x_{12}x_{21}x_{22}x_{13}x_{31}x_{23}x_{32}, \quad G_9 = x_{11}x_{12}x_{21}x_{22}x_{13}x_{31}x_{23}x_{32}x_{33},$$

connected by the syzygy

$$C_4D_5 = B_3E_6 = x_{11}^2x_{12}^2x_{21}^2x_{22}x_{13}x_{31}.$$

The associated fundamental partitions from which all others can be derived by addition are

$$A_1 = \begin{matrix} 1 & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix}, \quad B_3 = \begin{matrix} 1 & 1 & \bullet \\ 1 & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{matrix}, \quad C_4 = \begin{matrix} 1 & 1 & \bullet \\ 1 & 1 & \bullet \\ \bullet & \bullet & \bullet \end{matrix}, \quad D_5 = \begin{matrix} 1 & 1 & 1 \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{matrix},$$

$$E_6 = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & \bullet \\ 1 & \bullet & \bullet \end{matrix}, \quad F_8 = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \bullet \end{matrix}, \quad G_9 = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}.$$

These are connected by the simple syzygy

$$C_4 + D_5 = B_3 + E_6 = \begin{matrix} 2 & 2 & 1 \\ 2 & 1 & \bullet \\ 1 & \bullet & \bullet \end{matrix}.$$

The general form of xy-symmetrical plane partition of order 3 is therefore

$$\begin{matrix} A+B+C+D+E+F+G & B+C+D+E+F+G & D+E+F+G \\ B+C+D+E+F+G & C \quad +E+F+G & F+G \\ D+E+F+G & F+G & G \end{matrix}$$

where A, B, C, D, E, F, G are arbitrary positive integers.

519. The derived enumerating function is

$$\frac{1}{(1-x)(1-x^3)(1-x^5).(1-x^4)(1-x^6)(1-x^8)},$$

the numerator term disappearing.

Proceeding to restrict the part magnitude in the usual manner, we find the algebraic fraction

$$\frac{1-g^2x^9}{(1-g)(1-gx)(1-gx^3)(1-gx^4)(1-gx^5)(1-gx^6)(1-gx^8)(1-gx^9)},$$

in which we have to seek the coefficient of g^n.

This, with some little trouble, is found to be

$$\frac{(1-x^{n+1})(1-x^{n+3})(1-x^{n+5}).(1-x^{2n+4})(1-x^{2n+6})(1-x^{2n+8})}{(1-x)(1-x^3)(1-x^5).(1-x^4)(1-x^6)(1-x^8)}.$$

520. If we were to proceed to the next case, $i=4$, we should find the work very laborious; too much so to be given here in detail; but the result that would emerge would be

$$\frac{(1-x^{n+1})(1-x^{n+3})(1-x^{n+5})(1-x^{n+7}).(1-x^{2n+4})(1-x^{2n+6})(1-x^{2n+8})^2(1-x^{2n+10})(1-x^{2n+12})}{(1-x)(1-x^3)(1-x^5)(1-x^7).(1-x^4)(1-x^6)(1-x^8)^2(1-x^{10})(1-x^{12})}$$

Guided by the circumstance that for the ith order the generating function must reduce for $n=1$ to the function which enumerates self-conjugate partitions in the Theory of Line Partitions, we may construct the enumerating function for the order i. It is, written in Cayley's notation $1-x^s=(\mathbf{s})$, an algebraic fraction, of which the numerator is

$$\begin{array}{l} \mathbf{(n+1)\,(n+3)\,(n+5)\;\ldots\ldots\ldots\ldots\;(n+2i-1)} \\ \times\,\mathbf{(2n+4)\,(2n+6)\,(2n+8)\ldots\ldots\ldots\ldots(2n+4i-4)} \\ \times\,\mathbf{(2n+8)\,(2n+10)\,(2n+12)\;\ldots\ldots\ldots\;(2n+4i-8)} \\ \times\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \times\,\mathbf{(2n+4s)\,(2n+4s+2)\,(2n+4s+4)\;\ldots\ldots\;(2n+4i-4s)} \\ \times\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

wherein if i be even there are, omitting the first row, $\frac{1}{2}i$ rows the last of which is

$$\mathbf{(2n+2i)};$$

and if i be uneven there are $\frac{1}{2}(i-1)$ rows, the last of which is

$$\mathbf{(2n+2i-2)\,(2n+2i)\,(2n+2i+2)};$$

and the denominator is obtained from the numerator by putting $n=0$, viz. it is

$$\begin{array}{l} \mathbf{(1)\,(3)\,(5)\;\ldots\ldots\ldots\ldots\;(2i-1)} \\ \times\,\mathbf{(4)\,(6)\,(8)\;\ldots\ldots\ldots\ldots\;(4i-4)} \\ \times\,\mathbf{(8)\,(10)\,(12)\;\ldots\ldots\ldots\;(4i-8)} \\ \times\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \times\,\mathbf{(4s)\,(4s+2)\,(4s+4)\;\ldots\ldots\;(4i-4s)} \\ \times\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

the last row being $\mathbf{(2i)}$ or $\mathbf{(2i-2)\,(2i)\,(2i+2)}$ according as i is even or uneven.

521. The reader must be warned that, although there is little doubt that this result is correct, its truth at present rests upon the fact that it represents faithfully every particular case that has been examined, and that some consequences deduced from it also appear to be correct. The result has not been rigorously established. Nor has the expression when multiplied by g^n been summed into the form of an algebraic fraction. Further investigations in regard to these matters would be sure to lead to valuable work.

522. We proceed to deduce some consequences from the result.

When the number of layers of nodes is unrestricted we put $n=\infty$, and then the numerator becomes equal to unity. The function in that case may also be written

$$\frac{(1+x)(1+x^3)(1+x^5)\ldots\ldots\ldots\ldots\ldots\ldots(1+x^{2i-1})}{(2)(6)(10)\ldots(4i-2)\,.\,(4)(6)(8)^2(10)^2(12)^3(14)^3\ldots},$$

the last factors in the denominator being determined from the formula. Here the numerator is precisely the enumerating function of self-conjugate line partitions of at most i parts. It is, in fact, precisely what the whole expression, in the general formula, becomes when n is put equal to unity. It is at once seen to be

$$\frac{(2)(6)(10)\ldots(4i-2)}{(1)(3)(5)\ldots(2i-1)},$$

which is $$(1+x)(1+x^3)\ldots(1+x^{2i-1}).$$

523. On putting $n=2$ we obtain a very interesting result.

We find

$$\frac{(2i+1)}{(1)}\cdot\frac{(2i+4)(2i+6)\ldots(4i-2)(4i)}{(4)(6)\ldots(2i-2)(2i)},$$

which may be written

$$\frac{(2i+2)(2i+4)\ldots(4i-2)(4i)}{(2)(4)\ldots(2i-2)(2i)}$$

$$+x\frac{(2i+4)(2i+6)\ldots(4i-2)(4i)}{(2)(4)(6)\ldots(2i-2)}.$$

This expression enumerates plane partitions which possess the properties:

(i) they have xy-symmetry;

(ii) the part magnitude does not exceed 2;

(iii) there are not more than i parts in the direction of either axis.

For an even weight $2w$ we have to take the coefficient of x^{2w} in

$$\frac{(2i+2)(2i+4)\ldots(4i-2)(4i)}{(2)(4)\ldots(2i-2)(2i)},$$

or of x^w in $$\frac{(i+1)(i+2)\ldots(2i-1)(2i)}{(1)(2)\ldots(i-1)(i)}.$$

This function, as we have seen, enumerates in the Theory of Line Partitions those partitions of the number w in which both the part magnitude and the number of parts are limited so as not to exceed i.

524. Hence a correspondence between the plane partitions of $2w$, for which there is xy-symmetry, the part magnitude does not exceed two, and there are not more than i parts in the direction of either axis; and the line partitions of w, for which the part magnitude and the number of parts are both limited so as not to exceed i.

Or we may state the correspondence in another manner.

It is between the at most two layer xy-symmetrical graphs, of weight $2w$, restricted as to nodes along the x and y axes by the number i; and the graphs in two dimensions of weight w restricted as to nodes along the axes by the number i.

Ex. gr. for $w=4$, $i=3$, the correspondence is

111	11	11	|	221	22	111
1	11	1	|	2	22	111
		1	|	1		11

525. For an uneven weight $2w+1$ we similarly take the coefficient of x^w in

$$\frac{(i+2)(i+3)\dots(2i-1)(2i)}{(1)(2)(3)\dots(i-2)(i-1)},$$

and this enumerates partitions of w limited as to part magnitude by $i+1$, and as to number of parts by $i-1$.

526. Hence a correspondence between the plane partitions of $2w+1$, for which there is xy-symmetry, the part magnitude does not exceed two, and there are not more than i parts in the direction of either axis; and the line partitions of w, limited as to part magnitude by $i+1$, and as to number of parts by $i-1$.

As before, we have the alternative statement of the correspondence.

Ex. gr. for $w=5$, $i=4$, we have the five-to-five correspondence

11111	1111	111	111	11
	1	11	1	11
			1	1

1111	2111	2211	221	222
111	111	21	211	21
111	11	1	11	2
1	1	1		

When i is infinite the generating function becomes

$$\frac{1+x}{(1-x^2)(1-x^4)(1-x^6)\dots\text{ad inf.}}.$$

This result shews that the whole number of plane partitions of w which possess xy-symmetry and no part greater than two, is equal to the whole number of partitions of $\frac{1}{2}w$ or $\frac{1}{2}(w-1)$, according as w is even or uneven.

Ex. gr. Since the number 5 has seven line partitions we must have precisely that number of plane partitions, of the nature considered, of the number fifteen. These are

11111111	111111	11111	1111
1	111	1111	1111
1	111	111	1111
1	1	11	111
1	1	1	
1	1		
1			
1			

2111111	211111	21111
11	111	1111
1	11	11
1	1	11
1	1	1
1	1	
1		

527. There is one circumstance which gives a high probability to the conjectured enumerating function of xy-symmetrical graphs. It will have been remarked in a previous chapter that the function which enumerates plane partitions in which no particular symmetry is in question is composed of factors of the type $(n+s)/(s)$ when the corresponding lattice of nodes is complete. Further it was seen that one such factor being placed in regular order at each node the whole function was the product of such factors.

Now in the present case take, not a complete lattice of nodes, but that portion which involves the principal diagonal and all nodes to the right of it. If i is the number which limits the part magnitude and the number of parts, the successive rows of such a triangular lattice involve $i, i-1, i-2, \ldots 1$ nodes. The factors of the enumerating function can be placed at these nodes in the following manner:

$$\begin{array}{cccccc}
\frac{(n+1)}{(1)} & \frac{(2n+4)}{(4)} & \frac{(2n+6)}{(6)} & \frac{(2n+8)}{(8)} & \cdots & \frac{(2n+2i)}{(2i)} \\
 & \frac{(n+3)}{(3)} & \frac{(2n+8)}{(8)} & \frac{(2n+10)}{(10)} & \cdots & \frac{(2n+2i+2)}{(2i+2)} \\
 & & \frac{(n+5)}{(5)} & \frac{(2n+12)}{(12)} & \cdots & \frac{(2n+2i+4)}{(2i+4)} \\
 & & & \frac{(n+7)}{(7)} & \cdots & \frac{(2n+2i+6)}{(2i+6)} \\
 & & & & \ddots & \vdots \\
 & & & & & \frac{(2n+4i-4)}{(4i-4)} \\
 & & & & & \frac{(n+2i-1)}{(2i-1)}
\end{array}$$

Proceeding from the left-hand node always to the right or downwards till we arrive at (say) the sth node, we place a factor $\frac{(n+s)}{(s)}$ or $\frac{(2n+2s)}{(2s)}$ according as the node is on or not on the principal diagonal.

In xy-symmetry a node not on the diagonal represents (or may be taken to represent) the two nodes which are symmetrically situated in regard to the diagonal. In this case we see that the letters in the factor are doubled.

This property of the enumerating function is of great beauty and mathematical elegance.

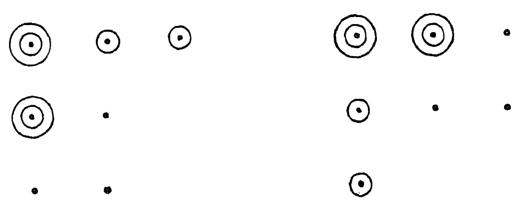
528. There is one more nature of symmetry that occurs in solid graphs. We have seen in a previous chapter (Art. 428) that graphs may have two aspects and be therefore capable of supplying two readings. Such occur for the first time at weight 13. They are in plane partition form

322	331
31	211
11	2

the first reading by rows as the second does by columns.

We have two distinct plane partitions associated with two solid graphs which are in reality the same graph under different aspects.

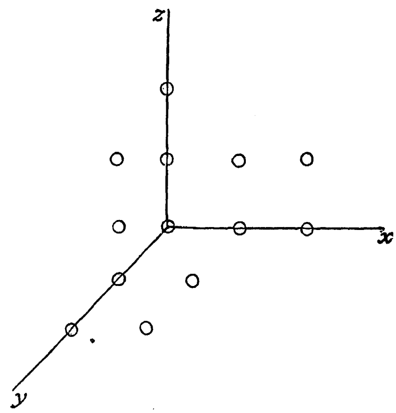
The graphs are



However they be viewed they invariably give one of these two aspects.

The fundamental property which is possessed by the corresponding plane partitions is this. The line partitions which constitute the successive rows are the conjugates of the line partitions which constitute the successive columns, and also the line partitions in question are not all self-conjugate, so that the plane partition does *not* read the same by rows as by columns. The latter condition is necessary, for if it were not complied with the plane partition would possess xyz-symmetry and the associated solid graph would have only one aspect.

We now construct the solid graph of the left hand of the two partitions above written. Projecting perpendicular to the plane xy we obtain the partition.



The reader will observe that every node is in one of the coordinate planes. This fact follows as a matter of course from the circumstance that in the partition the second number along the principal diagonal is less than 2. Further make the observation that the plane graphs in each of the three coordinate planes, if suitably read, are graphs of the line partition 322 which is the first row of the plane partition.

In the plane xy we read parallel to the axis of x,

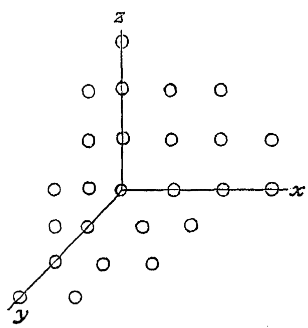
"	"	yz	"	"	"	y,
"	"	zx	"	"	"	z.

It thus appears that the whole plane partition is dependent upon the first row of parts.

This first row has a highest part which also designates the number of parts in the row. Moreover the first row partition is not self-conjugate. If we treat in this manner a partition having i parts and a highest part equal to i we obtain, so long as the partition is not self-conjugate, a plane partition associated with a solid graph which has precisely two aspects. To make this evident take the line partition

$$4332$$

and form a solid graph by placing the corresponding plane graph in each of the coordinate planes according to the above rule:



This solid graph denotes the plane partition

4332
411
311
11

and if we had started with the line partition 4431 we should have arrived at

4431
3111
311
2

It thence appears that we can at once enumerate all plane partitions which are derived from solid graphs which have:

(i) two aspects,

(ii) every node in one of the coordinate planes.

Because we can derive such a plane partition uniquely from every line partition which has the two properties:

(i) the highest part is equal to the number of parts,

(ii) it is not self-conjugate.

529. Consider the case where the highest part and number of parts are both equal to i.

If the highest part and number of parts are both conditioned not to exceed i the enumerating function is

$$\frac{(1)(2)\dots(2i)}{\{(1)(2)\dots(i)\}^2}.$$

Hence the enumerating function we require is

$$\frac{(1)(2)\dots(2i)}{\{(1)(2)\dots(i)\}^2} - 2\frac{(1)(2)\dots(2i-1)}{(1)(2)\dots(i)\,.\,(1)(2)\dots(i-1)} + \frac{(1)(2)\dots(2i-2)}{\{(1)(2)\dots(i-1)\}^2},$$

which reduces to $$x^{2i-1}\frac{(1)(2)\dots(2i-2)}{\{(1)(2)\dots(i-1)\}^2}.$$

We must subtract from this the function which enumerates the partitions which are self-conjugate. This is

$$x^{2i-1}(1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3}).$$

Hence the required enumerating function is

$$x^{2i-1}\left[\frac{(1)(2)\dots(2i-2)}{\{(1)(2)\dots(i-1)\}^2} - (1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3})\right],$$

and summing for i we obtain

$$\sum_{i=2}^{i=\infty} x^{2i-1}\left[\frac{(1)(2)\dots(2i-2)}{\{(1)(2)\dots(i-1)\}^2} - (1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3})\right].$$

530. The function enumerates *for a given weight of first row* all t plane partitions derived from solid graphs with two aspects and havi every node upon one of the coordinate planes.

Thus for $i=2$, the enumerator for a given value of i gives zero.

For $i=3$, we get

$$x^5\left\{\frac{(1)\,(2)\,(3)\,(4)}{(1)^2\,(2)^2}-(1+x)(1+x^3)\right\}$$

$$=x^5(1+x+2x^2+x^3+x^4-1-x-x^3-x^4)$$

$$=2x^7,$$

verifying the two line partitions (322), (331).

531. So also for $i=4$, we arrive at the expression

$$2(x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}) \quad \text{or} \quad 2x^9\frac{(6)}{(1)},$$

verifying the partitions

4221	4311	4222	4411	4322	4421
4332	4431	4333	4441	4433	4442

and establishing that each is the first row of a plane partition which enj the stated properties. These are in fact

4221	4222	4322
31	41	411
11	11	21
1	11	11

4332	4333	4433
411	411	4111
311	411	411
11	111	211

and the six others formed by taking the columns of these as rows.

532. Similarly for $i=5$ the function which enumerates the first rows found to be

$$2x^{11}\frac{(6)\,(9)}{(1)\,(2)}.$$

It will be clear on consideration that if w be the weight of the first r the weight of the associated plane partition is

$$3(w-i)+1.$$

We now derive the function which enumerates the plane partitions of given weight from that which enumerates the first rows of given weight by performing successively the operations:

(i) multiply by x^{-i},

(ii) write x^3 for x,

(iii) multiply by x,

and we obtain

$$\Sigma x^{3i-2}\left[\frac{(3)\,(6)\ldots(6i-6)}{\{(3)\,(6)\ldots(3i-3)\}^2}-(1+x^3)(1+x^9)\ldots(1+x^{6i-9})\right],$$

and in particular for $i=3$, 4, 5 the functions

$$2x^{13},\quad 2x^{16}\frac{(18)}{(3)},\quad 2x^{19}\frac{(18)\,(27)}{(3)\,(6)}\,*.$$

533. These plane partitions which possess the peculiar symmetry we are investigating form, from the graph point of view, solid angles since every node is upon one of the coordinate planes. Every partition must involve one of these solid angles in its graph. In other words these solid angles of nodes constitute the boundaries, so far as the coordinate planes are concerned, of the graphs of all of the partitions under examination. From these boundary partitions we can construct others by fitting into the solid angles, when possible, graphs which have either one or two aspects. Thus from the boundary partition

$$\begin{array}{l}322\\31\\11\end{array}$$

we can derive the partition

$$\begin{array}{l}322\\32\\11\end{array}$$

by fitting into the solid angle graph a single node which of course is a one-aspect graph. We of course could not fit in the next most simple graph, viz.:

$$\begin{array}{ll}\odot & \bullet\\ \bullet & \end{array}\equiv\begin{array}{l}21\\1\end{array}\text{ of one aspect,}$$

because that would make

$$\begin{array}{l}322\\331\\12\end{array}$$

which is not a partition at all.

* The general expression appears to be $2x^{3i+4}\dfrac{(18)}{(3)}\cdot\dfrac{(3i+12)\,(3i+13)\ldots(6i-3)}{(6)\,(9)\ldots(3i-9)}$.

So if we choose, of the order four, the boundary partition

4442
4111
3111
31

we can fit into the solid angle either of the graphs of nodes equivalent to the partitions

1	21	311	22	22	321	331	331
	1	1	21	22	21	211	221
		1			1	2	2

thus constructing the partitions

4442	4442	4442	4442
4211	4321	4422	4331
3111	3211	3211	3321.
31	31	32	31
4442	4442	4442	4442
4331	4432	4442	4442
3331	3321	3322	3332
31	32	33	33

each of which possesses the symmetry we are examining.

We thus see that the enumeration of the solid angle or boundary partitions is only the first step towards the enumeration of the partitions before us. The enumeration has not yet been carried any further. The relation of the graphs to a cube of i^3 nodes is the same as that which we found in the case of xyz-symmetrical graphs. The supplementary graph also has exactly two aspects. Thus from

322
31
11

we immediately deduce

322	4444
32	4433
11	4431
	4221

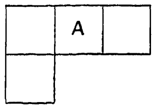
and an infinity of others.

This fact must be of importance in an investigation into the form of the complete enumerating function.

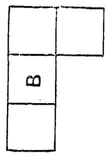
534. It is necessary to make some further remarks upon the different aspects of regular graphs in three dimensions. Consider first of all the graph in two dimensions of the partition (31)

• • •
•

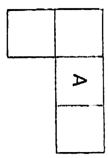
and construct it in another manner by means of four square plates fastened



together so as to form one piece. Suppose that the two faces of the piece are lettered A and B. Observe that the piece cannot be made to represent the conjugate partition (211) so long as the face A is exposed to view, but that if it be turned over so as to expose the face B we can place it in the required manner, viz.:

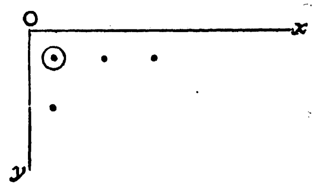


Because we can expose either face of the graph at pleasure the graphs are not essentially different. If we are restricted to the face A of the graph we can only represent the partition (211) by regarding the reflection of the graph last depicted about its left-hand edge as a proper representation. Thus we may agree to so regard the piece with the A face exposed as under:

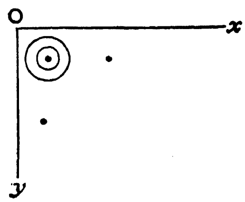


When we come to graphs in three dimensions we may similarly construct the graph in another manner by means of cubes fastened together. Instead of vertical piles of nodes we have vertical piles of cubes raised upon the plane xy. If the graph be also a graph in two dimensions we may treat the piece, composed of blocks all in one plane, as above, and, by turning it over, cause it to represent one, three, or six multipartite partitions as the case may be. Thus a single cube may be placed in one position; two cubes joined together in three positions, viz., with the greatest length along either of the three axes; the piece above depicted, when composed of four cubes, in six positions, viz., in two positions in each of the planes xy, yz, zx.

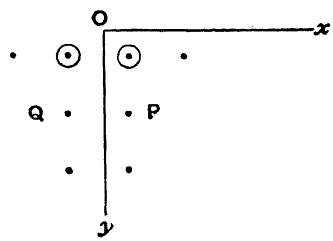
When however the graph is essentially three-dimensional the case is different. Consider the graph of the partition (31, 10) of the bipartite number (41):



where at the origin there is a pile of two nodes in the direction of the axis of z. If we replace the nodes by cubes and fasten them together we can place the piece in a second position:

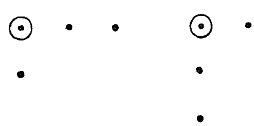


which is the graph of the partition (21, 10, 10), but we cannot place it in the third remaining position, which is the graph of the partition (211, 100). Turning the piece over is of no service, and we can only realise the missing position by adopting the first position, after rotation clockwise through a right angle,



as a proper representation of the partition. This graph Q is the reflection of the real graph P in the plane yz.

The two graphs



when formed of cubes fastened together are essentially different pieces, but they are reflections of one another.

In the theory of partitions discussed in this book they are regarded as being essentially graphs which are convertible the one into the other. We have two alternatives. We must either admit the principle of reflection, or we must denote the graphs by numbers written in space of two dimensions, so that they may be turned over. For example, we may denote the above graph by

$$\begin{array}{l} 211 \\ 1 \end{array}$$

so that we may turn it over and give it the position

$$\begin{array}{l} 21 \\ 1 \\ 1 \end{array}$$

The vertical pile of two nodes at the origin is regarded as a flat number two for the purpose in hand. It ceases to be so when we come to the graph

$$\begin{array}{l} 31 \\ 1 \end{array}$$

In every case the representation by nodes in three dimensions is replaced by a plane partition in two dimensions; this is then regarded as composed of square plates numbered on both faces so as to be reversible. At any moment, however, we may pass back to the representation *in solido*. Thus the two plane partitions

$$\begin{array}{lll} 322 & \quad & 331 \\ 31 & & 211 \\ 11 & & 2 \end{array}$$

are not essentially different when constructed of square plates numbered on both faces, but they are so when made up of cubes piled up on the plane xy; they are in that case not convertible the one into the other except by reflection.

In fact, in regard to the representation by cubes fastened together, every form, which is essentially three-dimensional, has either one or three aspects; every form, which is really two-dimensional, has one, three, or six aspects. The above two forms, for instance, have each only one aspect when composed of cubes; but when made up of square plates numbered on both faces they are identical, and each has two readings in the theory of partitions.

SECTION XI

SYMMETRIC FUNCTIONS OF SEVERAL SYSTEMS OF QUANTI-TIES, WITH SOME APPLICATIONS TO DISTRIBUTIO THEORY

CHAPTER I

ELEMENTARY THEORY

535. For certain parts of the Theory of Distributions a knowledge required of the Theory of the Symmetric Functions of several systems quantities.

It will be sufficiently illustrative of the general theory to consider mere two systems of quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3, \ldots,$$
$$\beta_1,\ \beta_2,\ \beta_3, \ldots .$$

For the single system we wrote

$$\begin{aligned}
&(1+\alpha_1 x)(1+\alpha_2 x)(1+\alpha_3 x)\ldots \\
&= 1 + \Sigma\alpha_1 . x + \Sigma\alpha_1\alpha_2 . x^2 + \Sigma\alpha_1\alpha_2\alpha_3 . x^3 + \ldots, \\
&= 1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \\
&= 1 + (1)\,x + (1^2)\,x^2 + (1^3)\,x^3 + \ldots,
\end{aligned}$$

and considered symmetric functions

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots = (p_1 p_2 p_3 \ldots).$$

So now we write

$$\begin{aligned}
&(1+\alpha_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y)(1+\alpha_3 x+\beta_3 y)\ldots \\
&= 1 + \Sigma\alpha_1 . x + \Sigma\beta_1 . y + \Sigma\alpha_1\alpha_2 . x^2 + \Sigma\alpha_1\beta_2 . xy + \Sigma\beta_1\beta_2 . y^2 + \ldots, \\
&= 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \ldots, \\
&= 1 + (10)x + (01)y + (10\,10)x^2 + (10\,01)xy + (01\,01)y^2 + (10^3)x^3 + (10^2\,01)x^2y +
\end{aligned}$$

The most general symmetric function before us is

$$\Sigma\alpha_1^{p_1}\beta_1^{q_1}\alpha_2^{p_2}\beta_2^{q_2}\alpha_3^{p_3}\beta_3^{q_3}\ldots,$$

enoted, symbolically, by

$$(p_1q_1\ p_2q_2\ p_3q_3\ \ldots).$$

The summation is in regard to the expressions obtained by permutation the n suffixes 1, 2, 3, ... but as a rule it is not necessary to specify the lue of n, and it may be regarded as being as large as may be necessary.

The generalization of the nomenclature and notation for three or more stems of quantities is evident.

The weight of the symmetric function is bipartite, and depends upon e numbers

$$p_1+p_2+p_3+\ldots=\Sigma p,$$
$$q_1+q_2+q_3+\ldots=\Sigma q.$$

We speak of the biweight $(\Sigma p, \Sigma q)$ of the symmetric function.

The expression $(p_1q_1\ p_2q_2\ p_3q_3\ \ldots)$ is a partition of the bipartite number $p, \Sigma q$. The sum $\Sigma p+\Sigma q$ is the weight of the number w if $\Sigma p+\Sigma q=w$. ssociated with any number w there is a biweight corresponding to every mposition of w into two parts, zero counting as a part.

Repetitions of biparts are denoted by exponents; thus

$$(p_1q_1^2)=(p_1q_1p_1q_1).$$

In the relation above

$$a_{pq}=(10^p\ 01^q),$$

d a_{pq} is an elementary symmetric function; it is also said to be "single-nitary," because each bipart of its partition contains but a single unit. The pression "unitary" is reserved for those symmetric functions whose partitions volve only units and zeros, such as the tripartite partition (111 110) for stance.

536. The number of partitions of the bipartite number pq is the co-ficient of x^py^q in the ascending expansion of the algebraic fraction

$$\frac{1}{1-x\,.\,1-y\,.\,1-x^2\,.\,1-xy\,.\,1-y^2\,.\,1-x^3\,.\,1-x^2y\,.\,1-xy^2\,.\,1-y^3\ldots};$$

r the general term of the expansion is

$$x^{m_{10}}y^{m_{01}}x^{2m_{20}}(xy)^{m_{11}}y^{2m_{02}}x^{3m_{30}}(x^2y)^{m_{21}}(xy^2)^{m_{12}}y^{3m_{03}}\ldots,$$

hich is equal to x^py^q if

$$m_{10}+2m_{20}+m_{11}+3m_{30}+2m_{21}+m_{12}+\ldots=p,$$
$$m_{01}+m_{11}+2m_{02}+m_{21}+2m_{12}+3m_{03}+\ldots=q;$$

d this will be the case if pq possesses the partition

$$(10^{m_{10}}01^{m_{01}}20^{m_{20}}11^{m_{11}}02^{m_{02}}30^{m_{30}}21^{m_{21}}12^{m_{12}}03^{m_{03}}\ldots).$$

The number of times $x^p y^q$ appears in the expansion, or in other words the coefficient of $x^p y^q$, is equal to the number of partitions of pq.

The number of partitions of pq into exactly μ parts is, by similar reasoning, equal to the coefficient of $a^\mu x^p y^q$ in the expansion of

$$\frac{1}{1-ax\,.\,1-ay\,.\,1-ax^2\,.\,1-axy\,.\,1-ay^2\,.\,1-ax^3\,.\,1-ax^2y\,.\,1-axy^2\,.\,1-ay^3\ldots}.$$

This function is, when expanded to a few terms and omitting terms which are simple powers of x or of y,

$$(a+a^2)\,xy+(a+2a^2+a^3)\,(x^2y+xy^2)+(a+3a^2+2a^3+a^4)\,(x^3y+xy^3)$$
$$+\,(a+4a^2+3a^3+a^4)\,x^2y^2+\ldots,$$

and the corresponding partitions are

Number 11	Number 21	Number 12
(11)	(21)	(12)
(10 01)	(20 01)	(10 02)
	(11 10)	(01 11)
	$(10^2\ 01)$	$(10\ 01^2)$

Number 31	Number 22	Number 13
(31)	(22)	(13)
(21 10)	(21 01)	(12 01)
(30 01)	(12 10)	(03 10)
(20 11)	(20 02)	(02 11)
(20 10 01)	(11^2)	(02 10 01)
$(11\ 10^2)$	$(20\ 01^2)$	$(11\ 01^2)$
$(10^3\ 01)$	$(02\ 10^2)$	$(10\ 01^3)$
	(11 10 01)	
	$(10^2\ 01^2)$	

537. The partitions with one bipart correspond to the sums of the powers in the single system or unipartite theory. We will express them in terms of the elementary functions. We will write (p_1q_1) in the form $s_{p_1q_1}$, just as with a single system we write (p_1) in the form s_{p_1}.

Taking logarithms of both sides of the relation

$$(1+\alpha_1x+\beta_1y)\,(1+\alpha_2x+\beta_2y)\ldots=1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots,$$

and remarking that

$$\Sigma\,(\alpha_1x+\beta_1y)=s_{10}x+s_{01}y,\quad \Sigma\,(\alpha_1x+\beta_1y)^2=s_{20}x^2+2s_{11}xy+s_{02}y^2,\text{ etc.,}$$

there results

$$(s_{10}x+s_{01}y)-\tfrac{1}{2}\,(s_{20}x^2+2s_{11}xy+s_{02}y^2)+\tfrac{1}{3}\,(s_{30}x^3+3s_{21}x^2y+3s_{12}xy^2+s_{03}y^3)+\ldots$$
$$=\log\,(1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots);$$

leading to
$$1 + a_{10}x + a_{01}y + \dots + a_{pq}x^p y^q + \dots$$
$$= \exp\{(s_{10}x + s_{01}y) - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2) + \dots\};$$

and now expanding the logarithm and exponential functions in these identities, we find by comparison of like powers of x and y:

$$\begin{cases} s_{10} = a_{10} \\ s_{01} = a_{01} \end{cases} \qquad \begin{cases} a_{10} = s_{10} \\ a_{01} = s_{01} \end{cases}$$

$$\begin{cases} s_{20} = a_{10}^2 - 2a_{20} \\ s_{11} = a_{10}a_{01} - a_{11} \\ s_{02} = a_{01}^2 - 2a_{02} \end{cases} \qquad \begin{cases} 2a_{20} = s_{10}^2 - s_{20} \\ a_{11} = s_{10}s_{01} - s_{11} \\ 2a_{02} = s_{01}^2 - s_{02} \end{cases}$$

$$\begin{cases} s_{30} = a_{10}^3 - 3a_{20}a_{10} + 3a_{30} \\ s_{21} = a_{10}^2a_{01} - a_{20}a_{01} - a_{11}a_{10} + a_{21} \\ s_{12} = a_{01}^2a_{10} - a_{02}a_{10} - a_{11}a_{01} + a_{12} \\ s_{03} = a_{01}^3 - 3a_{02}a_{01} + 3a_{03} \end{cases} \qquad \begin{cases} 6a_{30} = s_{10}^3 - 3s_{20}s_{10} + 2s_{30} \\ 2a_{21} = s_{10}^2s_{01} - s_{20}s_{01} - 2s_{11}s_{10} + 2s_{21} \\ 2a_{12} = s_{01}^2s_{10} - s_{02}s_{10} - 2s_{11}s_{01} + 2s_{12} \\ 6a_{03} = s_{01}^3 - 3s_{02}s_{01} + 2s_{03} \end{cases}$$

$$\vdots \qquad\qquad\qquad \vdots$$

and by an application of the multinomial theorem we find

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}s_{pq} = \sum_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots}a_{p_1q_1}^{\pi_1}a_{p_2q_2}^{\pi_2}\dots,$$

$$(-)^{p+q-1}a_{pq} = \sum_{\pi}\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!}\right\}^{\pi_2}\dots\frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\dots}s_{p_1q_1}^{\pi_1}s_{p_2q_2}^{\pi_2}\dots,$$

the summation, in each formula, being for all partitions

$$(p_1q_1^{\pi_1}\,p_2q_2^{\pi_2}\dots)$$

of the bipartite number (pq).

538. In the case of the single system we denoted the sum of the monomial symmetric functions of weight p by h_p. In the present instance, if h_{pq} denote the corresponding sum for the biweight pq, we find that

$$1 + h_{10}x + h_{01}y + h_{20}x^2 + h_{11}xy + h_{02}y^2 + \dots = \frac{1}{(1-\alpha_1x)(1-\alpha_2x)\dots(1-\beta_1y)(1-\beta_2y)\dots};$$

and, since the right-hand side is

$$(1 + h_{10}x + h_{20}x^2 + \dots)(1 + h_{01}y + h_{02}y^2 + \dots),$$

it is clear that
$$h_{pq} = h_{p0}h_{0q}.$$

538 *bis.* A more interesting function k_{pq} is defined by the relation

$$1 + a_{10}x + a_{01}y + \dots + a_{pq}x^py^q + \dots$$
$$= \frac{1}{1 - k_{10}x - k_{01}y + \dots + (-)^{p+q}k_{pq}x^py^q + \dots};$$

for then

$$1 + k_{10}x + k_{01}y + \ldots + k_{pq}x^p y^q + \ldots = \frac{1}{(1-\alpha_1 x - \beta_1 y)(1-\alpha_2 x - \beta_2 y)\ldots},$$

and
$$k_{pq} = \Sigma \binom{p_1+q_1}{p_1}\binom{p_2+q_2}{p_2}\ldots(p_1q_1\, p_2q_2\ldots).$$

Moreover,
$$1 + k_{10}x + k_{01}y + \ldots + k_{pq}x^p y^q + \ldots$$
$$= \frac{1}{1 - a_{10}x - a_{01}y + \ldots + (-)^{p+q}a_{pq}x^p y^q + \ldots},$$

so that, in any relation connecting the functions a_{pq} with the functions k_{pq}, we are always at liberty to interchange the symbols a and k.

We have the formula

$$(-)^{p+q-1}k_{pq} = \sum_{\pi} (-)^{\Sigma\pi - 1}\frac{(\Sigma\pi)!}{\pi_1!\,\pi_2!\ldots} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2}\ldots,$$

and the similar one with a and k interchanged.

From Art. 537

$$s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2) + \ldots$$
$$= -\log\{1 - k_{10}x - k_{01}y + \ldots + (-)^{p+q}k_{pq}x^p y^q + \ldots\},$$

and this, by the multinomial theorem, leads to

$$\frac{(p+q-1)!}{p!\,q!}s_{pq} = \sum_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}k_{p_1q_1}^{\pi_1}k_{p_2q_2}^{\pi_2}\ldots,$$

and it will be remarked that, if we multiply the left-hand side by $(-)^{p+q-1}$, we may replace the symbol k by the symbol a on the right-hand side.

Thus s_{pq} remains unchanged when k is replaced by a, except for a change of sign when $p+q$ is even.

CHAPTER II

THE THEORY OF SEPARATIONS AND THEORY OF DISTRIBUTION.

539. What has been said by way of definition an[d con]cerning the separations of a partition applies equally in [the case] of multipartite partitions. Thus of a partition $(p_1q_1\,p_2q_2\,p_3q_3)$ [a separation] might be $(p_1q_1\,p_3q_3)\,(p_2q_2)$ composed of separates $(p_1q_1\,p_3q_3)$ and (p_2q_2); its specification is

$$(p_1+p_3,\ q_1+q_3,\ p_2q_2),$$

a partition of the same bipartite number as the separated partition. The separation theory must now be connected with a Theory of Multipartite Distribution. Consider three identities

$$1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots=(1+\alpha_1'\,x+\beta_1'\,y)\,(1+\alpha_2'\,x+\beta_2'\,y)\ldots,$$
$$1+b_{10}x+b_{01}y+\ldots+b_{pq}x^py^q+\ldots=(1+\alpha_1''\,x+\beta_1''\,y)\,(1+\alpha_2''\,x+\beta_2''\,y)\ldots,$$
$$1+c_{10}x+c_{01}y+\ldots+c_{pq}x^py^q+\ldots=(1+\alpha_1'''x+\beta_1'''y)\,(1+\alpha_2'''x+\beta_2'''y)\ldots.$$

Assume the various coefficients of x and y in the two first identities to be given and those in the third to be connected with them by the relation

$$1+c_{10}\xi+c_{01}\eta+\ldots+c_{pq}\xi^p\eta^q+\ldots=\prod_s(1+\alpha_s'b_{10}\xi+\beta_s'b_{01}\eta+\ldots+\alpha'^p_s\beta'^q_s b_{pq}\xi^p\eta^q+\ldots),$$

ξ and η being undetermined quantities.

Multiplying out the right-hand side and denoting by partitions symmetric functions of the quantities $\alpha_1',\ \alpha_2',\ \alpha_3',\ldots;\ \beta_1',\ \beta_2',\ \beta_3',\ldots$ we find by comparison of coefficients of like powers of ξ and η the series of relations:

$$\begin{aligned}
c_{10}&=(10)\,b_{10},\\
c_{01}&=(01)\,b_{01},\\
c_{20}&=(20)\,b_{20}+(10^2)\,b_{10}^2,\\
c_{11}&=(11)\,b_{11}+(10\ 01)\,b_{10}\,b_{01},\\
c_{02}&=(02)\,b_{02}+(01^2)\,b_{01}^2,\\
c_{30}&=(30)\,b_{30}+(20\ 10)\,b_{20}\,b_{10}+(10^3)\,b_{10}^3,\\
c_{21}&=(21)\,b_{21}+(20\ 01)\,b_{20}\,b_{01}+(11\ 10)\,b_{11}\,b_{10}+(10^2\,01)\,b_{10}^2\,b_{01},\\
c_{12}&=(12)\,b_{12}+(02\ 10)\,b_{02}\,b_{10}+(11\ 01)\,b_{11}\,b_{01}+(01^2\,10)\,b_{01}^2\,b_{10},\\
c_{03}&=(03)\,b_{03}+(02\ 01)\,b_{02}\,b_{01}+(01^3)\,b_{01}^3,
\end{aligned}$$

etc.

$$c_{pq}=\Sigma\,(p_1q_1\,p_2q_2\,p_3q_3\ldots)\,b_{p_1q_1}\,b_{p_2q_2}\,b_{p_3q_3}\ldots,$$

the summation being for every partition of (pq).

It will be observed that, when the quantities b_{pq} are each of them put equal to unity, c_{pq} becomes h_{pq} and splits up into the product $h_{p0}h_{0q}$; moreover when $b_{10}=b_{01}=1$ and the remaining quantities b_{pq} are put equal to zero, c_{pq} becomes a_{pq}.

540. In order to establish comprehensive laws of symmetry in analogy with those appertaining to the unipartite case we now require a new theory of distributions.

Let parcels of type $(p_1q_1\,p_2q_2\,p_3q_3\,\ldots)$ be denoted by

$$A_1^{p_1}B_1^{q_1}A_2^{p_2}B_2^{q_2}A_3^{p_3}B_3^{q_3}\ldots,$$

that is to say there are

$$\begin{array}{lllll}
p_1 & \text{similar parcels each denoted by} & & & A_1, \\
q_1 & ,, & ,, & ,, & B_1, \\
p_2 & ,, & ,, & ,, & A_2, \\
q_2 & ,, & ,, & ,, & B_2, \\
p_3 & ,, & ,, & ,, & A_3, \\
q_3 & ,, & ,, & ,, & B_3, \\
\ldots & \ldots & \ldots & \ldots & \ldots;
\end{array}$$

let objects of type $(r_1s_1\,r_2s_2\,r_3s_3\,\ldots)$, where $\Sigma r=\Sigma p=n_1$, $\Sigma s=\Sigma q=n_2$, be denoted by

$$a_1^{r_1}b_1^{s_1}a_2^{r_2}b_2^{s_2}a_3^{r_3}b_3^{s_3}\ldots.$$

The number of objects is equal to the number of parcels and we define a distribution of the objects into the parcels, one object into each parcel, in the following manner:

"A distribution of objects of type $(r_1s_1\,r_2s_2\,r_3s_3\,\ldots)$ into parcels of type $(p_1q_1\,p_2q_2\,p_3q_3\,\ldots)$ is such that denoting the objects and parcels by

$$a_1^{r_1}b_1^{s_1}a_2^{r_2}b_2^{s_2}a_3^{r_3}b_3^{s_3}\ldots,\quad A_1^{p_1}B_1^{q_1}A_2^{p_2}B_2^{q_2}A_3^{p_3}B_3^{q_3}\ldots \text{ respectively,}$$

the objects $a_1^{r_1}a_2^{r_2}a_3^{r_3}\ldots$ are to be distributed into the parcels $A_1^{p_1}A_2^{p_2}A_3^{p_3}\ldots$ and the objects $b_1^{s_1}b_2^{s_2}b_3^{s_3}\ldots$ into the parcels $B_1^{q_1}B_2^{q_2}B_3^{q_3}\ldots$."

The number of these distributions is clearly equal to the product of two numbers, one denoting the number of ways of distributing objects of type $(r_1r_2r_3\ldots)$ into parcels of type $(p_1p_2p_3\ldots)$, and the other the number of ways of distributing objects of type $(s_1s_2s_3\ldots)$ into parcels of type $(q_1q_2q_3\ldots)$. From the unipartite theory the one number is given by the coefficient of $(r_1r_2r_3\ldots)_\alpha$ in the expansion of $h_{p_10}h_{p_20}h_{p_30}\ldots$ and the other by the coefficient of $(s_1s_2s_3\ldots)_\beta$ in the expansion of $h_{0q_1}h_{0q_2}h_{0q_3}\ldots$. Hence the number of distributions is given by the coefficient of

$$(r_1r_2r_3\ldots)_\alpha(s_1s_2s_3\ldots)_\beta,$$

in the expansion of $h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\ldots$, since $h_{pq}=h_{p0}h_{0q}$.

These distributions are considered to be unrestricted since the condition of the distribution that objects a and b must be placed in parcels A and B respectively is fundamental and does not count as a restriction.

We may write the unipartite functions $(r_1r_2r_3\ldots)_\alpha$, $(s_1s_2s_3\ldots)_\beta$ in the bipartite forms

$$(r_10\,r_20\,r_30\ \ldots),\ (0s_1\,0s_2\,0s_3\ldots),$$

and, multiplying them together, we see that the enumeration is given by the coefficient of the bipartite function $(r_1s_1\,r_2s_2\,r_3s_3\ldots)$ in the development of $h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\ldots$.

From the reciprocity that has been established in the unipartite case we see that we may interchange the partitions $(r_1r_2r_3\ldots)$, $(p_1p_2p_3\ldots)$ and also similarly the partitions $(s_1s_2s_3\ldots)$, $(q_1q_2q_3\ldots)$. Thus the enumeration is given in either of four forms:

(i) as coefficient of $(r_1s_1\ r_2s_2\ r_3s_3\ \ldots)$ in $h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\ldots$,

(ii) „ „ $(p_1s_1\ p_2s_2\ p_3s_3\ \ldots)$ in $h_{r_1q_1}h_{r_2q_2}h_{r_3q_3}\ldots$,

(iii) „ „ $(r_1q_1\ r_2q_2\ r_3q_3\ \ldots)$ in $h_{p_1s_1}h_{p_2s_2}h_{p_3s_3}\ldots$,

(iv) „ „ $(p_1q_1\,p_2q_2\,p_3q_3\ \ldots)$ in $h_{r_1s_1}h_{r_2s_2}h_{r_3s_3}\ldots$.

The last of these is derived at once from the bipartite distribution by a simple interchange of objects and parcels or of capital and small letters in the literal distribution scheme. The development of $h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\ldots$ enumerates the distribution of objects of all types into parcels of type $(p_1q_1\,p_2q_2\,p_3q_3\ \ldots)$ and is therefore the "Distribution Function," the distributions being unrestricted.

As in the unipartite case every distribution has itself a type specified by a partition of the original bipartite number. If, in the distribution, we observe

$$\text{objects } a_{s_1}^{\sigma_1}b_{s_1}^{\sigma_2} \text{ in parcels } A_{s_2}^{\sigma_1}B_{s_2}^{\sigma_2},$$

where s_1, s_2 are any integers and σ_1, σ_2 are integers one of which may be zero, we are given one part $\sigma_1\sigma_2$ of the partition which specifies the type of the distribution.

Ex. gr. For the distribution of objects of type (22) into parcels of type (11 10 01) we find only one distribution shewn by

$$\begin{array}{cccc} A_1 & B_1 & A_2 & B_3 \\ a_1 & b_1 & a_1 & b_1 \end{array}$$

Here for $s_1 = s_2 = 1$ we have $\sigma_1 = 1,\ \sigma_2 = 1$,

$s_1 = 1,\ s_2 = 2$ „ $\sigma_1 = 1,\ \sigma_2 = 0$,

$s_1 = 1,\ s_2 = 3$ „ $\sigma_1 = 0,\ \sigma_2 = 1$,

shewing that the distribution is of type (11 10 01).

If we interchange capital and small letters we find

$$\begin{matrix} A_1 & A_1 & B_1 & B_1 \\ a_1 & a_2 & b_1 & b_3 \end{matrix}$$

and we observe that the type of the distribution has not changed; it is still necessarily (11 10 01).

In general the interchange of capital and small letters does not alter the type of the distribution. This circumstance is of first-rate importance because it leads directly, as will be seen presently, to a theorem of algebraic symmetry.

The interchange of capital and small letters may be denoted by the substitutions

$$(Aa)(Bb).$$

Looking merely to the effect of the substitutions upon the type of the distribution we have

$$(Aa)(Bb)=1.$$

If we merely interchange the symbols A, a the types of both the objects and parcels connected with the distribution are changed and, in general, the type of the distribution is also changed. That this is so is seen at once from the definition; for when objects $a_{s_1}^{\sigma_1} b_{s_1}^{\sigma_2}$ occur in parcels $A_{s_2}^{\sigma_1} B_{s_2}^{\sigma_2}$, giving rise to a part $\sigma_1\sigma_2$ in the partition specifying the type, it is clear that if A, a be interchanged the part $\sigma_1\sigma_2$ does not arise unless $s_1=s_2$. Similarly also for the interchange of the symbols B, b.

We find however that the interchange (Aa) produces a distribution of the same type as the interchange (Bb). This is so because if we first make the substitution (Aa) and second the substitutions $(Aa)(Bb)$ the latter has no effect upon the type of distribution. Accordingly

$$(Aa)=(Aa)\,.\,(Aa)(Bb)=(Bb),$$

since $$(Aa)(Aa)=1.$$

The effect of (Aa) upon the type of the distribution is therefore the same as that of (Bb).

541. The distribution function $h_{p_1q_1}h_{p_2q_2}h_{p_3q_3}\dots$ is precisely the same as $c_{p_1q_1}c_{p_2q_2}c_{p_3q_3}\dots$ when the whole of the quantities b_{pq} are put equal to unity. If we retain these quantities b_{pq} and use the product $c_{p_1q_1}c_{p_2q_2}c_{p_3q_3}\dots$ we are able to single out the distributions which have any given type. For consider the relation

$$c_{21}c_{01}=(21)(01)b_{21}b_{01}+(2001)(01)b_{20}b_{01}^2+(1110)(01)b_{11}b_{10}b_{01}+(101001)(01)b_{10}^2b_{01}^2$$

and make the observations:

(i) the symmetric function product attached to any b-product is a separation of the partition which specifies the b-product;

(ii) the symmetric function product in every term is a separation of specification (21 01), (21 01) being the partition which specifies the c-product.

If we further develop the right-hand side of the relation we find

$$\begin{aligned} c_{21}c_{01} = &\{(22) + (21\ 01)\}\, b_{21}b_{01} + \{(21\ 01) + (20\ 02) + 2\,(20\ 01\ 01)\}\, b_{20}\, b_{01}^2 \\ &+ \{(12\ 10) + 2\,(11\ 11) + (11\ 10\ 01)\}\, b_{11}b_{10}b_{01} \\ &+ \{(11\ 10\ 01) + (10\ 10\ 02) + 2\ (10\ 10\ 01\ 01)\}\, b_{10}^2\, b_{01}^2. \end{aligned}$$

We will now exhibit the whole of the distributions involved.

Parcels of type (21 01).

Type of Objects	The Distribution	Type of Distribution
(22)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_1\ \ b_1$	(21 01)
(21 01)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_1\ \ b_2$	(21 01)
(21 01)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_2\ \ b_1$	$(20\ 01^2)$
(20 02)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_2\ \ b_2$	$(20\ 01^2)$
(20 01 01)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_2\ \ b_3$	$(20\ 01^2)$
(20 01 01)	$A_1A_1B_1\ B_2$ $a_1\ a_1\ b_3\ \ b_2$	$(20\ 01^2)$
(12 10)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_1\ \ b_1$	(11 10 01)
(11 11)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_1\ \ b_2$	(11 10 01)
(11 11)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_2\ \ b_1$	(11 10 01)
(11 10 01)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_1\ \ b_3$	(11 10 01)
(11 10 01)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_3\ \ b_1$	$(10^2\ 01^2)$
(10 10 02)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_3\ \ b_3$	$(10^2\ 01^2)$
(10 10 01 01)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_3\ \ b_4$	$(10^2\ 01^2)$
(10 10 01 01)	$A_1A_1B_1\ B_2$ $a_1\ a_2\ b_4\ \ b_3$	$(10^2\ 01^2)$

It will be observed that the type of the distribution is invariably the same as that which specifies the b-product to which the symmetric function, denoted by the same partition as the type of the objects, is attached. To explain this consider for a moment the terms

$$(21\ 01)\, b_{21}\, b_{01}, \quad (21\ 01)\, b_{20}\, b_{01}^2$$

in the expression of $c_{21}\, c_{01}$, and therewith the second and third distributions set out above. The term $(21\ 01)\, b_{21}\, b_{01}$ arises from the development of the term $(21)\,(01)\, b_{21}\, b_{01}$ and thus $(21\ 01)$ arises from the product $(21)\ (01)$; therefore it must be possible to distribute the objects of type $(21\ 01)$ in such wise that the type of the distribution is $(21\ 01)$. Similarly $(21\ 01)\, b_{20}\, b_{01}^2$ arises from the development of $(20\ 01)\ (01)\, b_{20}\, b_{01}^2$; $(21\ 01)$ comes from the product $(20\ 01)\,(01)$ and thus it must be possible to distribute objects of type $(21\ 01)$ in such wise that the type of the distribution is $(20\ 01\ 01)$.

Hence if

$$c_{p_1 q_1}\, c_{p_2 q_2}\, c_{p_3 q_3} \ldots = \ldots + \theta\,(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)\, b_{t_1 u_1}\, b_{t_2 u_2}\, b_{t_3 u_3} \ldots + \ldots,$$

there are θ ways of distributing objects of type $(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)$ into parcels of type $(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)$ so that the distributions are of type $(t_1 u_1\ t_2 u_2\ t_3 u_3 \ldots)$. We have shewn also that we may interchange the types of the objects and parcels without the type of the distribution being affected. Hence also

$$c_{r_1 s_1}\, c_{r_2 s_2}\, c_{r_3 s_3} \ldots = \ldots + \theta\,(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)\, b_{t_1 u_1}\, b_{t_2 u_2}\, b_{t_3 u_3} \ldots + \ldots.$$

542. We have therefore an intuitive proof of a law of symmetry. If we write

$$c_{p_1 q_1}\, c_{p_2 q_2}\, c_{p_3 q_3} \ldots = \ldots + P b_{t_1 u_1}\, b_{t_2 u_2}\, b_{t_3 u_3} \ldots + \ldots,$$

P is a linear function of separations of the function $(t_1 u_1\ t_2 u_2\ t_3 u_3 \ldots)$, each of which has a specification $(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)$. If P be developed so that

$$P = \ldots + \theta\,(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots) + \ldots,$$

we have seen that the number θ remains unchanged when the partitions $(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)$, $(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)$ are interchanged, while the partition $(t_1 u_1\ t_2 u_2\ t_3 u_3 \ldots)$ remains constant. Hence the whole of the functions $(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)$ that arise by multiplying out the various functions P must be the same as the various specifications of the separations of $(t_1 u_1\ t_2 u_2\ t_3 u_3 \ldots)$.

There is therefore a one-to-one correspondence between the functions P and the functions $(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)$, and we may form a table which possesses row and column symmetry, by expressing the different functions P appertaining to the various specifications $(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)$ of the separations of $(t_1 u_1\ t_2 u_2\ t_3 u_3 \ldots)$ in terms of the equi-numerous functions $(r_1 s_1\ r_2 s_2\ r_3 s_3 \ldots)$ which in some order are identical with the functions $(p_1 q_1\ p_2 q_2\ p_3 q_3 \ldots)$.

There is a symmetrical table in connexion with every partition of a given bipartite number. There will also be the inverse tables expressing the

monomial functions whose partitions are specifications of the separations in terms of the members of the system of functions, each member being a definite linear function of the separations which have a particular specification.

The inverse tables will also be symmetrical. This may be established by making use of the operators as in a previous chapter, or it may be regarded as necessarily following from the known properties of determinants.

In the direct tables the corresponding distributions are all of them of type $(t_1u_1\, t_2u_2\, t_3u_3 \ldots)$. When in particular the chosen partition is $(10^p\, 01^q)$ we find that the whole of the monomial functions are expressible in terms of the separations. Each separate is an elementary function, and the corresponding distributions are such that into a set of parcels $A_s^{\sigma_1} B_s^{\sigma_2}$ are placed objects, no two of which have the same suffix. The combinations $a_t a_t$, $a_t b_t$ are not permissible.

As in the unipartite case we have a theorem of expressibility, which may be enunciated as follows:

Theorem of Expressibility.

543. "If any partitions of the bipartite numbers $p_1q_1, p_2q_2, p_3q_3, \ldots$ be selected, and all of the parts assembled so as to form another partition of the same biweight as $(p_1q_1\, p_2q_2\, p_3q_3 \ldots)$, the symmetric function $(p_1q_1\, p_2q_2\, p_3q_3 \ldots)$ is expressible as a linear function of the separations of the partition thus formed."

There is also the theory of Groups of Separations, and in the inverse tables the sum of the coefficients of the separations of a group is zero, according to the theorem:

"In the expression of a symmetric function

$$(p_1q_1^{\pi_1}\, p_2q_2^{\pi_2}\, p_3q_3^{\pi_3} \ldots)$$

by means of separations of

$$(t_1u_1^{\tau_1}\, t_2u_2^{\tau_2}\, t_3u_3^{\tau_3} \ldots),$$

the algebraic sum of the coefficients *in each group* will be zero if the partition

$$(p_1q_1^{\pi_1}\, p_2q_2^{\pi_2}\, p_3q_3^{\pi_3} \ldots)$$

possesses no separations of specification

$$(\tau_1t_1,\ \tau_1u_1,\ \tau_2t_2,\ \tau_2u_2,\ \tau_3t_3,\ \tau_3u_3,\ \ldots)."$$

The law may be verified in the case of the table of separations of $(10^2\, 01^2)$ for the symmetric functions (22), $(21\ 01)$, $(12\ 10)$, (11^2), $(11\ 10\ 01)$, for none of these five functions can be separated with a specification $(20\ 02)$. On the other hand, the law does not obtain for the functions $(20\ 02)$, $(20\ 01^2)$, $(02\ 10^2)$, $(10^2\ 01^2)$, for each of these four functions *can* be separated with a specification $(20\ 02)$.

It must be observed that in the unipartite case there is no group theory when the replaced partition is (1^n), and the separates are elementary functions. In the bipartite case there is no such exception, *because* the separated partition $(10^p\,01^q)$ does not consist of repetitions of a single bipart. Hence the group theory exists when functions are expressed in terms of the elementary functions. On the other hand, an exception occurs when the separable partition *does* consist merely of a single bipart repeated.

Ex. gr. Exceptions occur for the separated partitions (11^2), (21^3), etc.

543 *bis*. In order to examine more closely the subject of multipartite separation we recall the identities

$$1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots = (1 + \alpha_1'\, x + \beta_1'\, y)(1 + \alpha_2'\, x + \beta_2'\, y)\ldots,$$
$$1 + b_{10}x + b_{01}y + \ldots + b_{pq}x^p y^q + \ldots = (1 + \alpha_1''\, x + \beta_1'' y)(1 + \alpha_2'' x + \beta_2''\, y)\ldots,$$
$$1 + c_{10}x + c_{01}y + \ldots + c_{pq}x^p y^q + \ldots = (1 + \alpha_1''' x + \beta_1''' y)(1 + \alpha_2''' x + \beta_2''' y)\ldots,$$

and the connecting relation

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots = \prod_s (1 + \alpha_s' b_{10}\xi + \beta_s' b_{01}\eta + \ldots + \alpha_s'^p \beta_s'^q b_{pq}\xi^p\eta^q + \ldots).$$

The right-hand side of this relation is, from the second of the three identities, if we put therein $x = \alpha_s'\xi$, $y = \beta_s'\eta$ in succession for all values of s, equal to

$$\prod_s \prod_t (1 + \alpha_s'\alpha_t''\xi + \beta_s'\beta_t''\eta),$$

so that $$\prod_s (1 + \alpha_s'''\xi + \beta_s'''\eta) = \prod_s \prod_t (1 + \alpha_s'\alpha_t''\xi + \beta_s'\beta_t''\eta),$$

or taking logarithms

$$\sum_s \log(1 + \alpha_s'''\xi + \beta_s'''\eta) = \sum_s \sum_t \log(1 + \alpha_s'\alpha_t''\xi + \beta_s'\beta_t''\eta).$$

We now expand each side in ascending powers of ξ and η, and observing that

$$\sum_s \sum_t (\alpha_s'\alpha_t'')^p (\beta_s'\beta_t'')^q = (pq)_1\,(pq)_2,$$

where symmetric functions of quantities α', β'; α'', β''; α''', β''', are denoted by partitions in brackets $(\ )_1$, $(\ )_2$, $(\ )_3$ respectively, we find by comparison of coefficients of $\xi^p\eta^q$ the noteworthy identity

$$(pq)_3 = (pq)_1\,(pq)_2.$$

This important result may be written in the two forms

$$\sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2}\ldots = (pq)_2 \sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2}\ldots,$$

$$\sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2}\ldots = (pq)_1 \sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2}\ldots.$$

In fact, the connecting relation is unaltered when the singly dashed quantities α', β', ... are severally interchanged with the doubly dashed quantities α'', β'', ...

Taking the latter form of the result the next step will be to compare the coefficient of the term

$$b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots,$$

on the dexter with the coefficient of the same term when the sinister is developed by means of the relation

$$c_{pq} = \Sigma\,(p_1q_1\,p_2q_2\,p_3q_3\ldots)\,b_{p_1q_1}b_{p_2q_2}b_{p_3q_3}\ldots,$$

of Art. 539.

When the left-hand side is multiplied out each symmetric function product which multiplies the term

$$b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots$$

is necessarily a separation of the symmetric function

$$(p_1q_1^{\pi_1}\,p_2q_2^{\pi_2}\ldots).$$

The result of the comparison will be therefore the expression of the function $(pq)_1$ in terms of such separations.

Let $$\frac{(-)^{\Sigma\rho-1}(\Sigma\rho-1)!}{\rho_1!\,\rho_2!\ldots}\,c_{r_1s_1}^{\rho_1}\,c_{r_2s_2}^{\rho_2}\ldots$$

be a term of the left-hand side of the identity.

Since the coefficient herein of

$$b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots$$

is a linear function of separations of

$$(p_1q_1^{\pi_1}\,p_2q_2^{\pi_2}\ldots)$$

of specification $$(r_1s_1^{\rho_1}\,r_2s_2^{\rho_2}\ldots),$$

it is clear that we are only concerned with those terms in the expression of

$$c_{r_1s_1},\quad c_{r_2s_2},\ldots$$

which involve symmetric functions which are separates of separations of

$$(p_1q_1^{\pi_1}\,p_2q_2^{\pi_2}\ldots).$$

In other words, we need only to attend to the terms which involve exclusively symmetric functions composed of the parts p_1q_1, p_2q_2, ...

If we do this there is no necessity to attend to the b-products at all.

Let us denote then by

$$(J_1),\ (J_2),\ \ldots\ (J_1'),\ (J_2'),\ \ldots\ \ldots\ldots$$

various separates of separations of

$$(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots),$$

and suppose that the effective part of $c_{r_1 s_1}$ is

$$(J_1) + (J_2) + \ldots,$$

and of $c_{r_2 s_2}$

$$(J_1') + (J_2') + \ldots,$$

etc.

Then the effective part of

$$\frac{(-)^{\Sigma\rho-1}(\Sigma\rho-1)!}{\rho_1!\,\rho_2!\ldots} c_{r_1 s_1}^{\rho_1} c_{r_2 s_2}^{\rho_2} \ldots$$

is

$$\frac{(-)^{\Sigma\rho-1}(\Sigma\rho-1)!}{\rho_1!\,\rho_2!\ldots} \Sigma \frac{\rho_1!}{\rho_{11}!\,\rho_{12}!\ldots} (J_1)^{\rho_{11}} (J_2)^{\rho_{12}} \ldots \frac{\rho_2!}{\rho_{21}!\,\rho_{22}!\ldots} (J_1')^{\rho_{21}} (J_2')^{\rho_{22}} \ldots\ \ldots\ldots$$

wherein of course

$$\rho_{11} + \rho_{12} + \ldots = \rho_1, \quad \rho_{21} + \rho_{22} + \ldots = \rho_2, \ldots.$$

This may be written

$$\Sigma \frac{(-)^{\Sigma\rho-1}(\Sigma\rho-1)!}{\rho_{11}!\,\rho_{12}!\ldots\rho_{21}!\,\rho_{22}!\ldots\ \ldots\ldots} (J_1)^{\rho_{11}} (J_2)^{\rho_{12}} \ldots (J_1')^{\rho_{21}} (J_2')^{\rho_{22}} \ldots\ \ldots\ldots,$$

wherein

$$\sum_s \rho_{1s} + \sum_s \rho_{2s} + \ldots = \rho;$$

and the symmetric function product

$$(J_1)^{\rho_{11}} (J_2)^{\rho_{12}} \ldots (J_1')^{\rho_{21}} (J_2')^{\rho_{22}} \ldots$$

is a separation of $(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots)$

of specification $(r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$.

From this result, in order to proceed to the effective part of

$$\sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1 q_1}^{\pi_1} c_{p_2 q_2}^{\pi_2} \ldots$$

we must consider all terms of the sum which involve a c-product

$$c_{r_1 s_1}^{\rho_1} c_{r_2 s_2}^{\rho_2} \ldots,$$

$(r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$ being a partition of the bipartite number pq, and also a specification of some separation of

$$(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots).$$

We find that the required effective part is necessarily

$$\sum_j \frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots,$$

where the summation is in respect of *every separation*

$$(J_1)^{j_1}(J_2)^{j_2}\dots$$

of the partition $$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2}\dots).$$

Hence the formula

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots}(pq)=\Sigma_j\frac{(-)^{\Sigma j}}{}$$

where the summation is in regard
$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2}\dots)$, and to every separat

544. This important result is a ge
for the expression of the sums of the pow
in terms of the elementary symmetric functions.
dual character because there are two systems of qu
is in terms of separations of an arbitrary partition of the biparti
The like reasoning at once extends the theorem to any number o
of quantities, and the reader will have no difficulty in realizing the truth of
the formula

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots}(pqr\dots)=\Sigma_j\frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\dots}(J_1)^{j_1}(J_2)^{j_2}\dots,$$

where the summation is for every separation

$$(J_1)^{j_1}(J_2)^{j_2}\dots$$

of the partition $(p_1 q_1 r_1\dots^{\pi_1}\,p_2 q_2 r_2\dots^{\pi_2}\dots)$ of the multipartite number

$$(pqr\dots).$$

545. Ex. gr. To express (31) by means of separations of (20 10 01).

Here $$\pi_1=\pi_2=\pi_3=1,\quad \Sigma\pi=3,$$

and we find

$$2(31)=(20\ 10\ 01)-(20\ 10)(01)-(20\ 01)(10)-(10\ 01)(20)+2(20)(10)(01).$$

Again to express (333) by means of separations of (311 011 011).

Here $\pi_1=1$, $\pi_2=2$, $\Sigma\pi=3$, and

$$(333)=(311\ 011\ 011)-(311\ 011)(011)-(011^2)(311)+(311)(011)^2.$$

546. The formula that has been reached may be written in the form

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots}S_{(p_1q_1^{\pi_1}p_2q_2^{\pi_2}\dots)}=\Sigma_j\frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\dots}(J_1)^{j_1}(J_2)^{j_2}\dots,$$

the notation being such that

$$S_{(p_1q_1^{\pi_1}p_2q_2^{\pi_2}\dots)}$$

denotes S_{pq} or (pq) when expressed in terms of separations of the symmetric function

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots).$$

In this notation, recalling a previous formula, viz.:

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} S_{pq} = \sum_{\pi} (-)^{\Sigma\pi-1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2! \ldots} a_{p_1 q_1}^{\pi_1} a_{p_2 q_2}^{\pi_2} \ldots,$$

since $a_{p_1 q_1} = (10^{p_1}\, 01^{q_1})$ and the formula expresses S_{pq} as a linear function of separations of the symmetric function

$$(10^p\, 01^q),$$

the formula is equivalent to

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} S_{(10^p\, 01^q)} = \sum_{j} (-)^{\Sigma j-1} \frac{(\Sigma j - 1)!}{j_1!\, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots.$$

CHAPTER III

THE DIFFERENTIAL OPERATIONS

547. The elegant properties of these symmetric functions are most easily established by means of the differential operations which we now consider. The theory is completely parallel to that of the single system discussed in Section I.

Starting with the identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \dots (1 + \alpha_n x + \beta_n y)$$
$$= 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \dots + a_{pq}x^p y^q + \dots,$$

where n may be as large as we please, and multiplying each side by $(1 + \mu x + \nu y)$, the right-hand side becomes

$$1 + (a_{10} + \mu)\,x + (a_{01} + \nu)\,y + (a_{20} + \mu a_{10})\,x^2 + (a_{11} + \mu a_{01} + \nu a_{10})\,xy + \dots,$$

and in general a_{pq} becomes converted into

$$a_{pq} + \mu a_{p-1,\,q} + \nu a_{p,\,q-1}.$$

Write now

$$g_{10} = \partial_{a_{10}} + a_{01}\partial_{a_{11}} + \dots + a_{p-1,\,q}\,\partial_{a_{pq}} + \dots,$$
$$g_{01} = \partial_{a_{01}} + a_{10}\partial_{a_{11}} + \dots + a_{p,\,q-1}\,\partial_{a_{pq}} + \dots.$$

Then any rational integral function of $a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \dots, a_{pq}, \dots$ say

$$f(a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \dots a_{pq}, \dots) = f,$$

becomes $\quad f(a_{10} + \mu,\ a_{01} + \nu,\ \dots,\ a_{pq} + \mu a_{p-1,\,q} + \nu a_{p,\,q-1}, \dots)$

which is by Taylor's Theorem

$$f + (\mu g_{10} + \nu g_{01}) f + \frac{1}{2!}(\mu g_{10} + \nu g_{01})^2 f + \frac{1}{3!}(\mu g_{10} + \nu g_{01})^3 f + \dots = \overline{\exp}\,(\mu g_{10} + \nu g_{01}).f,$$

where the bar of exp denotes that the multiplication of operators is symbolic as in Taylor's Theorem.

Moreover
$$\frac{1}{s!}(\mu g_{10} + \nu g_{01})^s = \Sigma \frac{1}{p!\,q!}(g_{10}^p\, g_{01}^q)\,\mu^p \nu^q,$$

the summation being subject to $p + q = s$ and the placing of $g_{10}^p g_{01}^q$ in a single bracket denotes symbolic multiplication of the operators.

Therefore if we write

$$\frac{1}{p\,!\,q\,!}(g_{10}^p g_{01}^q) = G_{pq},$$

we obtain

$$(1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f$$
$$= \overline{\exp}\,(\mu g_{10} + \nu g_{01}) \,.\, f.$$

Observe that as regards quantity we may write

$$1 + \mu a_{10} + \nu a_{01} + \mu^2 a_{20} + \mu\nu a_{11} + \nu^2 a_{02} + \ldots + \mu^p \nu^q a_{pq} + \ldots = \overline{\exp}\,(\mu s_{10} + \nu s_{01})$$

if we agree that on the right-hand side there is a symbolism according to which

$$\frac{1}{p\,!\,q\,!}(s_{10}^p s_{01}^q) \equiv a_{pq}.$$

This is important from the point of view by which there is throughout this subject an exact correspondence up to a certain point between the algebras of quantities and differential operations.

We are about to see that in these algebras

$$a_{pq},\ s_{pq}$$

correspond to $G_{pq},\ g_{pq}$ respectively.

548. Now suppose that the function f is a symmetric function which when expressed in terms of $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots$ is

$$(p_1 q_1\, p_2 q_2\, p_3 q_3 \ldots).$$

The introduction of the new quantities μ, ν results in the addition to

$$(p_1 q_1\, p_2 q_2\, p_3 q_3 \ldots)$$

of the new terms

$$\mu^{p_1}\nu^{q_1}(p_2 q_2\, p_3 q_3 \ldots) + \mu^{p_2}\nu^{q_2}(p_1 q_1\, p_3 q_3 \ldots) + \mu^{p_3}\nu^{q_3}(p_1 q_1\, p_2 q_2 \ldots) + \ldots.$$

Hence

$$(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f$$
$$= f + \mu^{p_1}\nu^{q_1}(p_2 q_2\, p_3 q_3 \ldots) + \mu^{p_2}\nu^{q_2}(p_1 q_1\, p_3 q_3 \ldots) + \ldots,$$

and equating coefficients of like products $\mu^p \nu^q$ we find

$$G_{rs} f = 0,$$

unless the bipart rs occurs in the expression of f,

$$G_{p_1 q_1}(p_1 q_1\, p_2 q_2\, p_3 q_3 \ldots) = (p_2 q_2\, p_3 q_3 \ldots),$$
$$G_{p_2 q_2}(p_1 q_1\, p_2 q_2\, p_3 q_3 \ldots) = (p_1 q_1\, p_3 q_3 \ldots),$$
$$G_{pq}(pq) = 1,$$
$$G_{p_1 q_1} G_{p_2 q_2} G_{p_3 q_3} \ldots (p_1 q_1\, p_2 q_2\, p_3 q_3 \ldots) = 1.$$

From these relations we gather the important fact that the effect of G_{pq} upon a symmetric function symbolized by a partition is to obliterate *one* part pq when such part is present and to cause the function to vanish when no part pq appears in it.

549. Returning to the relation of operators, now written without specification of an operand,

$$1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots = \overline{\exp} \, (\mu g_{10} + \nu g_{01})$$

we recall that on the right-hand side the multiplications of operators are symbolic. We now seek to replace the right-hand side by an expression which involves products of linear operators which are not symbolic products but on the contrary denote successive operations.

To the two linear operators

$$g_{10} = \partial_{a_{10}} + a_{10}\partial_{a_{20}} + a_{01}\partial_{a_{11}} + \ldots$$

$$g_{01} = \partial_{a_{01}} + a_{01}\partial_{a_{02}} + a_{10}\partial_{a_{11}} + \ldots$$

we add a general linear operator including them and defined by the statement

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,\,q}} + a_{01}\partial_{a_{p,\,q+1}} + \ldots + a_{rs}\partial_{a_{p+r,\,q+s}} + \ldots.$$

We will establish the relation

$$\overline{\exp} \, (m_{10}g_{10} + m_{01}g_{01} + m_{20}g_{20} + m_{11}g_{11} + \ldots + m_{pq}g_{pq} + \ldots)$$

$$= \exp \, (M_{10}g_{10} + M_{01}g_{01} + M_{20}g_{20} + M_{11}g_{11} + \ldots + M_{pq}g_{pq} + \ldots),$$

where if ξ, η be undetermined algebraic quantities, the numerical magnitudes M_{pq} are given in terms of the numerical magnitudes m_{pq} by the relation

$$\exp \, (M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p \eta^q + \ldots) = 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots.$$

First observe the well-known relation

$$(g_{pq})\,(g_{rs}) = (g_{pq}\,g_{rs}) + g_{p+r,\,q+s}$$

where on the left-hand side the multiplication denotes successive operations and the second term on the right-hand side is obtained by operating with g_{pq} upon g_{rs} where g_{rs} is considered to be a function of algebraic quantities only and the differential inverses are not subject to operation. The reader should verify the simple case

$$(g_{10})\,(g_{01}) = (g_{10}g_{01}) + g_{11}.$$

Put $$u_1 = m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots$$

and observe that if $m_{pq}g_{pq}$ be written symbolically $m_{pq}g$, we obtain

$$u_1 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)\, g.$$

Two successive operations of u_1, written $(u_1)^2$, result in two terms, one of which is an operator of the second order, written (u_1^2), and the other a linear operator, formed by simple differentiation as above explained, which we will write u_2:

$$(u_1)^2 = (u_1^2) + u_2.$$

Also $u_2 = \Sigma m_{pq} m_{rs} g_{p+r, q+s}$, and if we write $m_{pq} m_{rs} g_{p+r, q+s}$ in a symbolic form

$$m_{pq} m_{rs} g$$

it is at once seen that

$$u_2 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^2 g.$$

Similarly $$(u_1)(u_2) = (u_1 u_2) + u_3,$$

where $$u_3 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^3 g;$$

leading to $$u_s = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^s g,$$

and $$(u_s)(u_t) = (u_s u_t) + u_{s+t}.$$

Moreover it has been shewn in Volume I that for any series of linear partial differential operators $u_1, u_2, u_3, \ldots$ enjoying the property

$$(u_s)(u_t) = (u_s u_t) + u_{s+t},$$

we have $$\overline{\exp}\,(u_1) = \exp\,(u_1 - \tfrac{1}{2}u_2 + \tfrac{1}{3}u_3 - \ldots),$$

the multiplications which arise on the development of the right-hand side being non-symbolic.

Hence on substitution for $u_1, u_2, u_3, \ldots$

$$\overline{\exp}\,(m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots)$$
$$= \exp\,\{(m_{10} + m_{01} + \ldots)\,g - \tfrac{1}{2}(m_{10} + m_{01} + \ldots)^2 g + \tfrac{1}{3}(m_{10} + m_{01} + \ldots)^3 g - \ldots\}.$$

If the right-hand side be written in the form

$$\exp\,(M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots)$$

we must have, for undetermined multipliers ξ, η,

$$\exp\,\{(m_{10}\xi + m_{01}\eta + \ldots) - \tfrac{1}{2}(m_{10}\xi + m_{01}\eta + \ldots)^2 + \tfrac{1}{3}(m_{10}\xi + m_{01}\eta + \ldots) - \ldots\}$$
$$= \exp\,(M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots);$$

and since the left-hand side is

$$\exp\log\,(1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots),$$

we have the identity

$$\exp\,(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots)$$
$$= 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots$$

for the determination of the magnitudes M_{pq} when the magnitudes m_{pq} are given and *vice versâ*.

The relation is clearly otherwise written

$$M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots$$
$$= \log\,(1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots).$$

Comparing Art. 537 it is evident that m_{pq} and M_{pq} are connected in the same way as

$$a_{pq} \text{ and } (-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} s_{pq}.$$

Thence the formulæ of the same Art. give

$$M_{pq} = \Sigma (-)^{\Sigma\pi-1} \frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} m^{\pi_1}_{p_1q_1} m^{\pi_2}_{p_2q_2} \ldots,$$

$$m_{pq} = \Sigma \frac{1}{\pi_1!\,\pi_2!\ldots} M^{\pi_1}_{p_1q_1} M^{\pi_2}_{p_2q_2} \ldots,$$

the summation in each case being for all partitions

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$$

of the bipartite number pq.

The particular case of this theorem that we require now is obtained by putting $m_{10}=\mu$, $m_{01}=\nu$, $m_{pq}=0$ in other cases.

Thus

$$\overline{\exp}\,(\mu g_{10}+\nu g_{01}) = \exp\,(M_{10}g_{10}+M_{01}g_{01}+\ldots+M_{pq}g_{pq}+\ldots),$$

where $$M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots = \log(1+\mu\xi+\nu\eta).$$

We derive the relation

$$M_{pq} = (-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\mu^p\nu^q,$$

so that $$\overline{\exp}\,(\mu g_{10}+\nu g_{01})$$

$$=\exp\{(\mu g_{10}+\nu g_{01})-\tfrac{1}{2}(\mu^2g_{20}+2\mu\nu g_{11}+\nu^2g_{02})+\tfrac{1}{3}(\mu^3g_{30}+3\mu^2\nu g_{21}+3\mu\nu^2g_{12}+\nu^3g_{03})+\ldots\}.$$

Thence from Art. 547

$$(\mu g_{10}+\nu g_{01})-\tfrac{1}{2}(\mu^2g_{20}+2\mu\nu g_{11}+\nu^2g_{02})+\tfrac{1}{3}(\mu^3g_{30}+3\mu^2\nu g_{21}+3\mu\nu^2g_{12}+\nu^3g_{03})-\ldots$$
$$=\log(1+\mu G_{10}+\nu G_{01}+\ldots+\mu^p\nu^qG_{pq}+\ldots).$$

Expansion of the right-hand side, and subsequent comparison of like terms in μ, ν, yields the relations:

$g_{10}=G_{10}$	$G_{10}=g_{10}$
$g_{01}=G_{01}$	$G_{01}=g_{01}$
$g_{20}=G_{10}^2-2G_{20}$	$2G_{20}=g_{10}^2-g_{20}$
$g_{11}=G_{10}G_{01}-G_{11}$	$G_{11}=g_{10}g_{01}-g_{11}$
$g_{02}=G_{01}^2-2G_{02}$	$2G_{02}=g_{01}^2-g_{02}$
$g_{30}=G_{10}^3-3G_{20}G_{10}+3G_{30}$	$6G_{30}=g_{10}^3-3g_{20}g_{10}+2g_{30}$
$g_{21}=G_{10}^2G_{01}-G_{20}G_{01}-G_{11}G_{10}+G_{21}$	$2G_{21}=g_{10}^2g_{01}-g_{20}g_{01}-2g_{11}g_{10}+2g_{21}$
$g_{12}=G_{01}^2G_{10}-G_{02}G_{10}-G_{11}G_{01}+G_{12}$	$2G_{12}=g_{01}^2g_{10}-g_{02}g_{10}-2g_{11}g_{01}+2g_{12}$
$g_{03}=G_{01}^3-3G_{02}G_{01}+3G_{03}$	$6G_{03}=g_{01}^3-3g_{02}g_{01}+2g_{03}$
etc.	etc.

relations to be compared with those of Art. 537.

The corresponding general formulæ are

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}=\sum_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}G^{\pi_1}_{p_1q_1}G^{\pi_2}_{p_2q_2}\ldots,$$

$$(-)^{p+q-1}G_{pq}=\Sigma\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!}\right\}^{\pi_2}\ldots\frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\ldots}(g_{p_1q_1})^{\pi_1}(g_{p_2q_2})^{\pi_2}\ldots,$$

the summation being for every partition

$$(p_1q_1^{\pi_1}\,p_2q_2^{\pi_2}\ldots)$$

of the bipartite number pq.

In the second formula the multiplications of linear operators denote their successive performance.

550. By comparison of these relations with the corresponding algebraic ones to which reference has been made, it is found that g_{pq} and G_{pq} are in co-relation with s_{pq} and a_{pq} respectively. In other words, these operations correspond to the partitions (pq) and $(10^p\,01^q)$. It is necessary to find the operations which correspond to the remaining partitions which symbolize symmetric functions. We easily derive the relations

$$\begin{cases}(g_{p_1q_1})(g_{p_2q_2})=(g_{p_1q_1}\,g_{p_2q_2})+g_{p_1+p_2,\,q_1+q_2}\,,\\ \qquad (g_{pq})^2=(g^2_{pq})+g_{2p,\,2q}\,,\end{cases}$$

$$\begin{cases}(g_{p_1q_1})(g_{p_2q_2})(g_{p_3q_3})=(g_{p_1q_1}\,g_{p_2q_2}\,g_{p_3q_3})+(g_{p_1q_1}\,g_{p_2+p_3,\,q_2+q_3})+(g_{p_2q_2}\,g_{p_1+p_3,\,q_1+q_3})\\ \qquad\qquad +(g_{p_3q_3}\,g_{p_1+p_2,\,q_1+q_2})+g_{p_1+p_2+p_3,\,q_1+q_2+q_3}\,,\\ (g_{p_1q_1})^2(g_{p_2q_2})=(g^2_{p_1q_1}\,g_{p_2q_2})+2(g_{p_1q_1}\,g_{p_1+p_2,\,q_1+q_2})+(g_{2p_1,\,2q_1}\,g_{p_2q_2})\\ \qquad\qquad +g_{2p_1+p_2,\,2q_1+q_2}\,,\\ (g_{pq})^3=(g^3_{pq})+3(g_{2p,\,2q}\,g_{pq})+g_{3p,\,3q}\,;\end{cases}$$

and comparison with the algebraic formulæ

$$\begin{cases}(p_1q_1)(p_2q_2)=(p_1q_1\,p_2q_2)+(p_1+p_2,\,q_1+q_2),\\ \qquad (pq)^2=2(pq^2)+(2p,\,2q),\end{cases}$$

$$\begin{aligned}(p_1q_1)(p_2q_2)(p_3q_3)&=(p_1q_1\,p_2q_2\,p_3q_3)+(p_1q_1\,p_2+p_3,\,q_2+q_3)+(p_2q_2\,p_1+p_3,\,q_1+q_3)\\ &\quad+(p_3q_3\,p_1+p_2,\,q_1+q_2)+(p_1+p_2+p_3,\,q_1+q_2+q_3),\\ (p_1q_1)^2(p_2q_2)&=2(p_1q_1^2\,p_2q_2)+2(p_1q_1\,p_1+p_2,\,q_1+q_2)+(2p_1\,2q_1\,p_2q_2)\\ &\quad+(2p_1+p_2,\,2q_1+q_2),\\ (pq)^3&=6(pq^3)+3(2p,\,2q\,pq)+(3p,\,3q),\end{aligned}$$

shews that the operators

$$\frac{1}{2!}(g^2_{pq}),\quad \frac{1}{2!}(g^2_{p_1q_1}\,g_{p_2q_2}),\quad \frac{1}{3!}(g^3_{pq})$$

are produced according to the same law as the symmetric functions

$$(pq^2),\quad (p_1q_1^2\,p_2q_2),\quad (pq^3).$$

In general the operator

$$\frac{1}{\pi_1 !\, \pi_2 ! \ldots}$$

corresponds to the symmetric functio

$$(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots).$$

There is thus complete correspondence between quantity and operation, and any formula of quantity may be at once translated into a formula of operations.

Observe that a product of symmetric functions

$$(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots)\, (r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$$

is in correspondence with the operation

$$\frac{1}{\pi_1 !\, \pi_2 ! \ldots} (g_{p_1 q_1}^{\pi_1}\, g_{p_2 q_2}^{\pi_2} \ldots) \frac{1}{\rho_1 !\, \rho_2 ! \ldots} (g_{r_1 s_1}^{\rho_1}\, g_{r_2 s_2}^{\rho_2} \ldots),$$

the notation indicating that the two operations

$$\frac{1}{\pi_1 !\, \pi_2 ! \ldots} (g_{p_1 q_1}^{\pi_1}\, g_{p_2 q_2}^{\pi_2} \ldots) \text{ and } \frac{1}{\rho_1 !\, \rho_2 ! \ldots} (g_{r_1 s_1}^{\rho_1}\, g_{r_2 s_2}^{\rho_2} \ldots)$$

are to be successively performed.

Ex. gr. From the formula

$$(31\ 01) = -\tfrac{1}{2}(21\ 10)(01) + \tfrac{1}{2}(21\ 01)(10) + \tfrac{1}{2}(10\ 01)(21) - \tfrac{1}{2}(21\ 10\ 01)$$

we derive

$$(g_{31}\, g_{01}) = -\tfrac{1}{2}(g_{21}\, g_{10})(g_{01}) + \tfrac{1}{2}(g_{21}\, g_{01})(g_{10}) + \tfrac{1}{2}(g_{10}\, g_{01})(g_{21}) - \tfrac{1}{2}(g_{21}\, g_{10}\, g_{01}).$$

551. Since every symmetric function

$$(p_1 q_1^{\pi_1}\, p_2 q_2^{\pi_2} \ldots)$$

is expressible in terms of the elementary functions a_{pq}, we may express every operator

$$\frac{1}{\pi_1 !\, \pi_2 ! \ldots} (g_{p_1 q_1}^{\pi_1}\, g_{p_2 q_2}^{\pi_2} \ldots)$$

in terms of the operators G_{pq}.

The operation therefore may be performed upon a function symbolized by a partition by simply operating independently with the successive G-products and adding together the results.

As a particular result observe that since

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} g_{pq} = \Sigma (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1 !\, \pi_2 ! \ldots} G_{p_1 q_1}^{\pi_1}\, G_{p_2 q_2}^{\pi_2} \ldots,$$

we have
$$g_{pq} = (-)^{p+q-1} \frac{p!\, q!}{(p+q-1)!} G_{pq} + \ldots,$$

$$g_{pq}\, s_{pq} = (-)^{p+q-1} \frac{p\,!\,q\,!}{(p+q-1)\,!};$$

a theorem of symmetry established in Art. 542,

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots) = (-)^{\Sigma\pi + p + q} \frac{p\,!\,q\,!}{(p+q-1)\,!} \frac{(\Sigma\pi - 1)\,!}{\pi_1\,!\,\pi_2\,!\ldots} a_{pq} + \ldots,$$

we derive the result

$$(g^{\pi_1}_{p_1 q_1} g^{\pi_2}_{p_2 q_2} \ldots)\, s_{pq} = (-)^{\Sigma\pi + p + q} \frac{p\,!\,q\,!}{(p+q-1)\,!} (\Sigma\pi - 1)\,!\,.$$

Referring to the three identities of Art. 543 *bis*, and the relation

$$(pq)_3 = (pq)_1\,(pq)_2,$$

we see that in the relation

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots$$
$$= \prod_s (1 + \alpha_s' b_{10}\xi + \beta_s' b_{01}\eta + \ldots + \alpha'^p_s \beta'^q_s b_{pq}\xi^p\eta^q + \ldots)$$

we can interchange the singly dashed quantities α_s', β_s' with the doubly dashed quantities α_s'', β_s''.

Let the differential operators

$$\begin{array}{ll} g_{pq}' & G_{pq}' \\ g_{pq}'' & G_{pq}'' \\ g_{pq}''' & G_{pq}''' \end{array}$$

refer to the sets of quantities with one, two, and three dashes respectively and write the relation

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots$$
$$= \prod_s (1 + \alpha_s' b_{10}\xi + \beta_s' b_{01}\eta + \ldots + \alpha'^p_s \beta'^q_s b_{pq}\xi^p\eta^q + \ldots)$$

in the abbreviated form

$$U = u'_{\alpha_1\beta_1}\, u'_{\alpha_2\beta_2}\, u'_{\alpha_3\beta_3} \ldots.$$

The performance of the operation

$$g_{pq}'' = \partial_{b_{pq}} + b_{10}\partial_{b_{p+1,q}} + b_{01}\partial_{b_{p,q+1}} + \ldots$$

upon both sides of the relation yields

$$g_{pq}''\, U = (g_{pq}''\, u'_{\alpha_1\beta_1})\, u'_{\alpha_2\beta_2}\, u'_{\alpha_3\beta_3} + \ldots;$$

and since
$$g_{pq}''\, u'_{\alpha_s\beta_s} = \alpha'^p_s \beta'^q_s\, \xi^p\eta^q u'_{\alpha_s\beta_s},$$

it follows that
$$g_{pq}''\, U = (pq)_1 \xi^p\eta^q\, U.$$

If we now substitute $1 + c_{10}\xi + c_{01}\eta + \ldots$ for U, and compare coefficients, we find that

$$g_{pq}''\, c_{pq} = (pq)_1,$$
$$g_{pq}''\, c_{rs} = (pq)_1\, c_{r-p,\, s-q}.$$

We can now connect the operators g_{pq}'', g_{pq}''', for regarding the coefficients $_{pq}$ as functions of the coefficients c_{pq} only

$$\begin{aligned} g_{pq}'' &= (g_{pq}''\, c_{pq})\, \partial_{c_{pq}} + \ldots + (g_{pq}'\, c_{rs})\, \partial_{c_{rs}} + \ldots \\ &= (pq)_1 \left(\partial_{c_{pq}} + c_{10}\, \partial_{c_{p+1,\, q}} + c_{01}\, \partial_{c_{p,\, q+1}} + \ldots + c_{r-p,\, s-q}\, \partial_{c_{rs}} + \ldots\right) \\ &= (pq)_1\, g_{pq}'''. \end{aligned}$$

Similarly, or as a result of the symmetry that has been established,

$$g_{pq}' = (pq)_2\, g_{pq}''',$$

nd since $(pq)_3 = (pq)_2\, (pq)_1$, we may write down the relations

$$(pq)_1\, g_{pq}' = (pq)_2\, g_{pq}'' = (pq)_3\, g_{pq}'''.$$

If then we regard the relation

$$\begin{aligned} &1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots \\ &= \prod_s \left(1 + \alpha_s' b_{10}\xi + \beta_s' b_{01}\eta + \ldots\right) \\ &= \prod_s \left(1 + \alpha_s'' a_{10}\xi + \beta_s'' a_{01}\eta + \ldots\right) \end{aligned}$$

s defining a transformation of the set of quantities

$$c_{10},\ c_{01},\ \ldots\ c_{pq}\ \ldots$$

nto *either* of the sets of quantities

$$b_{10},\ b_{01},\ \ldots\ b_{pq}\ \ldots$$

$$a_{10},\ a_{01},\ \ldots\ a_{pq}\ \ldots,$$

he operation $\qquad (pq)_3\, g_{pq}'''$

s an invariant of the transformation.

Expressing the operators g in terms of the operators G by Art. 549, we btain the system of relations

$$\begin{aligned} G_{10}'' &= (10)_1\, G_{10}''' \\ G_{01}'' &= (01)_1\, G_{01}''' \\ G_{20}'' &= (20)_1\, G_{20}''' + (10^2)_1\, G_{10}^{2\,'''} \\ G_{11}'' &= (11)_1\, G_{11}''' + (10\ 01)_1\, G_{10}'''\, G_{01}''' \\ G_{02}'' &= (02)_1\, G_{02}''' + (01^2)_1\, G_{01}^{2\,'''} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

which should be compared with the relations set forth in Art. 539 between he quantities c_{pq} and b_{pq}.

553. In general we may take the following view:

We have

$$\begin{aligned} &\log\left(1 + \xi G_{10}'' + \eta G_{01}'' + \ldots + \xi^p\eta^q G_{pq}'' + \ldots\right) \\ &= \xi g_{10}'' + \eta g_{01}'' - \tfrac{1}{2}\left(\xi^2 g_{20}'' + 2\xi\eta g_{11}'' + \eta^2 g_{02}''\right) + \ldots \\ &= \sum_s \log\left(1 + \alpha_s' G_{10}'''\xi + \beta_s' G_{01}'''\eta + \ldots + \alpha_s'^p \beta_s'^q G_{pq}'''\xi^p\eta^q + \ldots\right); \end{aligned}$$

by making use of the relations

$$g_{pq}'' = (pq)_1 \, g_{pq}''',$$

we find $$\log(1 + \xi G_{10}''' + \eta G_{01}''' + \ldots + \xi^p \eta^q G_{pq}''' + \ldots)$$
$$= \xi g_{10}''' + \eta g_{01}''' - \tfrac{1}{2}(\xi^2 g_{20}''' + 2\xi\eta g_{11}''' + \eta^2 g_{02}''') + \ldots.$$

Hence $$1 + \xi G_{10}'' + \eta G_{01}'' + \ldots + \xi^p \eta^q G_{pq}'' + \ldots$$
$$= \prod_s (1 + \alpha_s' G_{10}''' \xi + \beta_s' G_{01}''' \eta + \ldots + \alpha_s'^p \beta_s'^q G_{pq}''' \xi^p \eta^q + \ldots),$$

and comparing this with the relation of Art. 539, viz.

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p \eta^q + \ldots$$
$$= \prod_s (1 + \alpha_s' b_{10}\xi + \beta_s' b_{01}\eta + \ldots + \alpha_s'^p \beta_s'^q b_{pq} \xi^p \eta^q + \ldots),$$

we arrive at the following important theorem:

"In any relation connecting the quantities c_{pq} with the quantities b_{pq} we can obtain a relation between operators by substituting G_{pq}'', G_{pq}''' for c_{pq}, b_{pq} respectively."

It is established in the same manner that in any relation connecting the quantities c_{pq} with the quantities a_{pq} we can obtain a relation between operators by substituting G_{pq}', G_{pq}''' for c_{pq}, a_{pq} respectively.

In fact, ξ, η being undetermined quantities in the assumed relation which connects the magnitudes in the three identities, we are able to express any product, or sum of products, of the quantities $c_{10}, c_{01}, \ldots c_{pq}, \ldots$ in terms of products of the quantities $b_{10}, b_{01}, \ldots b_{pq}, \ldots$ and of symmetrical functions of the quantities $\alpha_1', \beta_1', \alpha_2', \beta_2', \ldots$—or in terms of products of the quantities $a_{10}, a_{01}, \ldots a_{pq}, \ldots$ and of symmetrical functions of the quantities α_1'', β_1'', $\alpha_2'', \beta_2'', \ldots$. The substitutions before us can be carried out in any equation thus formed.

It must be borne in mind that in any relation connecting the three sets of quantities

$$\alpha_s', \beta_s'; \quad \alpha_s'', \beta_s''; \quad \alpha_s''', \beta_s''';$$

we may interchange the two sets

$$\alpha_s', \beta_s'; \quad \alpha_s'', \beta_s''.$$

We can now establish various laws of symmetry in regard to symmetric functions of several systems of quantities.

554. In view of the equality

$$(pq)_3 = (pq)_1 (pq)_2,$$

it is clear that any integral symmetric function of the system

$$\alpha_s''', \beta_s''',$$

which is expressible in brackets $(\ )_3$, can be expressed as a linear function of products of symmetric functions, each of which is of the form

$$(\ )_1\ (\ )_2,$$

and that in that expression we can interchange the brackets $(\ )_1$, $(\ )_2$.

We may therefore suppose an identity

$$\text{(A)} \qquad (r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)_3$$
$$= \ldots + J\{(a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots)_1 (p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_2 + (a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots)_2 (p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_1\},$$

wherein the functions

$$(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots), \quad (a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots), \quad (p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)$$

are all of the same weight.

Moreover, any product of the quantities $c_{10}, c_{01}, \ldots c_{pq}, \ldots$ can be expressed as a linear function of expressions, each of which contains a monomial symmetric function of the quantities α_s', β_s', and a product of the quantities $b_{10}, b_{01}, \ldots b_{pq}, \ldots$.

Assume then the relations

$$\text{(B)} \quad c_{p_1 q_1}^{\pi_1} c_{p_2 q_2}^{\pi_2} \ldots = \ldots + L(a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots)_1 b_{r_1 s_1}^{\rho_1} b_{r_2 s_2}^{\rho_2} \ldots + \ldots;$$

$$\text{(C)} \quad c_{a_1 b_1}^{\epsilon_1} c_{a_2 b_2}^{\epsilon_2} \ldots = \ldots + M(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_1 b_{r_1 s_1}^{\rho_1} b_{r_2 s_2}^{\rho_2} \ldots + \ldots.$$

It will be shewn that the numerical coefficients J, L, M are equal to one another.

For from the relation (B) is derived the operator relation

$$G''^{\pi_1}_{p_1 q_1} G''^{\pi_2}_{p_2 q_2} \ldots = \ldots + L(a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots)_1 G'''^{\rho_1}_{r_1 s_1} G'''^{\rho_2}_{r_2 s_2} \ldots + \ldots,$$

and performing each side upon the opposite side of relation (A) we obtain after cancelling $(a_1 b_1^{\epsilon_1} a_2 b_2^{\epsilon_2} \ldots)_1$,

$$L\, G'''^{\rho_1}_{r_1 s_1} G'''^{\rho_2}_{r_2 s_2} \ldots (r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)_3 = J\, G''^{\pi_1}_{p_1 q_1} G''^{\pi_2}_{p_2 q_2} \ldots (p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_2,$$

no other terms surviving the operations.

Hence $$L = J.$$

In a similar manner from the relations (A) and (C) we prove that

$$M = J.$$

As a particular verification observe the relations

$$(10^3\, 01)_3 = \ldots + 3\{(20\, 11)_1 (10^3\, 01)_2 + (10^3\, 01)_1 (20\, 11)_2\} + \ldots,$$
$$c_{20}\, c_{11} = \ldots + 3\,(10^3\, 01)\, b_{10}^3\, b_{01} + \ldots,$$
$$c_{10}^3\, c_{01} = \ldots + 3\,(20\, 11)\, b_{10}^3\, b_{01} + \ldots.$$

We have, therefore, another proof of the Law of Symmetry which was established in Art. 542 by the method of Distributions.

Since it has been shewn above that in any relation connecting the quantities c_{pq} with the quantities b_{pq} we are at liberty to substitute G_{pq}'', G_{pq}''' for c_{pq}, b_{pq} respectively, it follows from Art. 550 that in any relation which connects the monomial symmetric function

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_3$$

with a linear function of products of the quantities b_{pq} we are at liberty to substitute

$$\frac{1}{\pi_1 ! \, \pi_2 ! \ldots} (g''^{\pi_1}_{p_1 q_1} g''^{\pi_2}_{p_2 q_2} \ldots)$$

for

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)$$

and G_{pq}''' for b_{pq}.

555. This fact leads at once to a second law of symmetry, for if we derive, from the three initial identities and the relation assumed to exist between the quantities involved, a relation

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_3 = \ldots + P \, b^{\rho_1}_{r_1 s_1} b^{\rho_2}_{r_2 s_2} \ldots + \ldots,$$

wherein P is some symmetric function of the first system of quantities α_s', β_s', we are led to an operator relation

$$\frac{1}{\pi_1 ! \, \pi_2 ! \ldots} (g''^{\pi_1}_{p_1 q_1} g''^{\pi_2}_{p_2 q_2} \ldots) = \ldots + P \, G'''^{\rho_1}_{r_1 s_1} G'''^{\rho_2}_{r_2 s_2} \ldots + \ldots.$$

Now assume a second relation

$$(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)_3 = \ldots + Q b^{\pi_1}_{p_1 q_1} b^{\pi_2}_{p_2 q_2} \ldots + \ldots,$$

and utilise the operator relation in the manner

$$(\ldots + P \, G'''^{\rho_1}_{r_1 s_1} G'''^{\rho_2}_{r_2 s_2} \ldots + \ldots)(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)_3$$

$$= \frac{1}{\pi_1 ! \, \pi_2 ! \ldots} (g''^{\pi_1}_{p_1 q_1} g''^{\pi_2}_{p_2 q_2} \ldots)(\ldots + Q b^{\pi_1}_{p_1 q_1} b^{\pi_2}_{p_2 q_2} \ldots + \ldots),$$

and we find $P = Q$, since no other terms survive the operation.

We thus have before us

A Theorem of Symmetry.

"If $$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots)_3 = \ldots + P \, b^{\rho_1}_{r_1 s_1} b^{\rho_2}_{r_2 s_2} \ldots + \ldots,$$

the co-factor symmetric function P is unaltered when the partitions

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots), \quad (r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)$$

are interchanged."

The function P presents itself, in the first place, as a linear function of separations of the partition appertaining to the b-product to which it is

attached. The theorem furnishes linear functions of separations of any two partitions

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots), \quad (r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)$$

respectively of the same biweight, which are algebraically equal to one another.

To elucidate the matter we can form a table of biweight 21 as follows:

	b_{21}	$b_{20} b_{01}$	$b_{11} b_{10}$	$b_{10}^2 b_{01}$
$(21)_3$	$+(21)$	$+(20\,01)$ $-(20)(01)$	$+(11\,10)$ $-(11)(10)$	$+(10^2\,01)$ $-(10^2)(01)$ $-(10\,01)(10)$ $+(10)^2(01)$
$(20\,01)_3$	$-(21)$	$-(20\,01)$ $-(20)(01)$	$-(11\,10)$ $+(11)(10)$	$-(10^2\,01)$ $-(10^2)(01)$ $+(10\,01)(10)$
$(11\,10)_3$	$-(21)$	$-(20\,01)$ $+(20)(01)$	$-(11\,10)$	$-(10^2\,01)$ $+(10^2)(01)$
$(10^2\,01)_3$	$+(21)$	$+(20\,01)$	$+(11\,10)$	$+(10^2\,01)$

which is to be read by rows.

The table possesses certain properties:

(i) Each term in a column is a separation of the partition of the b-product at the head of the column.

(ii) The separations in any line of terms *as written* possess the same specification.

(iii) A row of terms possesses at most four lines, because the partitions prefixed to the rows, viz. (21), (20 01), (11 10), (10^2 01), are four in number (the same of course as the number of rows). In any row of terms the specifications of the separations in the first, second, third and fourth lines are equal respectively to the partitions (21), (20 01), (11 10), (10^2 01).

(iv) The separations which appear in the same line are affected by the same coefficients.

(v) The Theorem of Symmetry shews that algebraically the table possesses row and column symmetry.

The block of terms in the rth row and cth column is algebraically equal to the block of terms in the cth row and rth column.

A block of terms in the rth row and cth column may be briefly and suggestively defined to be

"A linear function of separations of the partition of the b-product at the head of the cth column formed according to the law of the function prefixed to the rth row."

We may then state the Theorem of Symmetry in the form: "The assemblage of separations of

$$(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots),$$

formed according to the law of the function

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots),$$

is algebraically equal to the assemblage of separations of

$$(p_1 q_1^{\pi_1} p_2 q_2^{\pi_2} \ldots),$$

formed according to the law of the function

$$(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)."$$

CHAPTER IV

THE LINEAR PARTIAL DIFFERENTIAL OPERATORS OF THE THEORY OF SEPARATIONS AND THE PARTITION OBLITERATING OPERATORS

556. We have seen that if the biparts of a monomial symmetric function θ are themselves partitioned *in any manner* into biparts so that when they are all assembled in a single bracket they are represented by the partition

$$(r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$$

the symmetric function θ is expressible as a linear function of certain assemblages of separations of the symmetric function $(r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$. We have moreover actually so expressed the functions whose partitions comprise but a single bipart. For the calculation of other functions in terms of separations of a determined function it is necessary to find the adaptation of the linear operators $g_{10}, g_{01}, \ldots g_{pq}, \ldots$ so that they may be performed on a function when it is expressed in terms of separations. Of any partition $(r_1 s_1^{\rho_1}\, r_2 s_2^{\rho_2} \ldots)$ separates are formed by taking all possible combinations of its parts one, two, etc. at a time. There are thus $(\rho_1 + 1)(\rho_2 + 1) \ldots - 1$ distinct separates which must be regarded as independent variables. Let any such separate be

$$(10^{\pi_{10}+\rho_{10}}\, 01^{\pi_{01}+\rho_{01}} \ldots rs^{\pi_{rs}+\rho_{rs}} \ldots),$$

where
$$(10^{\pi_{10}}\, 01^{\pi_{01}} \ldots rs^{\pi_{rs}} \ldots)$$

is some partition of (pq).

Then by a known theorem of transformation in the differential calculus

$$g_{pq} \equiv \Sigma \left\{ g_{pq} (10^{\pi_{10}+\rho_{10}}\, 01^{\pi_{01}+\rho_{01}} \ldots rs^{\pi_{rs}+\rho_{rs}} \ldots) \right\} \partial_{(10^{\pi_{10}+\rho_{10}}\, 01^{\pi_{01}+\rho_{01}} \ldots rs^{\pi_{rs}+\rho_{rs}} \ldots)},$$

where $\dfrac{d}{dx}$ has been written in the notation ∂_x and the summation on the right-hand side is in regard to every distinct separate of the partition $(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \ldots)$. Unless however a separate contains some partition of (pq) the operation of g_{pq} will cause it to vanish. We can immediately transform the expression of g_{pq} because

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} g_{pq} = \Sigma \frac{(-)^{\Sigma\pi - 1} (\Sigma\pi - 1)!}{\pi_{10}!\, \pi_{01}! \ldots \pi_{rs}! \ldots} G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \ldots G_{rs}^{\pi_{rs}} \ldots,$$

and

$$G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \dots G_{rs}^{\pi_{rs}} \dots (10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots) = (10^{\rho_{10}} 01^{\rho_{01}} \dots rs^{\rho_{rs}} \dots).$$

Hence
$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq}$$

$$= \sum_{\pi} \sum_{\rho} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} (10^{\rho_{10}} 01^{\rho_{01}} \dots rs^{\rho_{rs}} \dots)\, \partial_{(10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots)},$$

the summation being in regard to (i) every partition $(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)$ of (pq), (ii) every separate $(10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots)$ of the partition $(r_1 s_1^{\rho_1} r_2 s_2^{\rho_2} \dots)$. The right-hand side of this relation may be regarded as the sum of a number of portions in each of which the numbers $\pi_{10}, \pi_{01}, \dots \pi_{rs}, \dots$ are constant.

In fact writing

$$\sum_{\rho} (10^{\rho_{10}} 01^{\rho_{01}} \dots rs^{\rho_{rs}} \dots)\, \partial_{(10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots)} = g_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)},$$

we have

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq} = \sum_{\pi} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} g_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)};$$

a formula which should be compared with the two

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq} = \sum_{\pi} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \dots G_{rs}^{\pi_{rs}} \dots;$$

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} (pq) = \sum_{\pi} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} a_{10}^{\pi_{10}} a_{01}^{\pi_{01}} \dots a_{rs}^{\pi_{rs}} \dots;$$

the summation in each formula being for every partition $(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)$ of the bipartite number (pq).

The operator $g_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)}$ is of biweight pq. It is equal to

$$\sum_{\rho} (10^{\rho_{10}} 01^{\rho_{01}} \dots rs^{\rho_{rs}} \dots)\, \partial_{(10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots)},$$

where $(10^{\pi_{10}+\rho_{10}} 01^{\pi_{01}+\rho_{01}} \dots rs^{\pi_{rs}+\rho_{rs}} \dots)$ is a separate of the given separable partition. If this separable partition be written in the form

$$(10^{\sigma_{10}} 01^{\sigma_{01}} \dots rs^{\sigma_{rs}} \dots),$$

the expression of $g_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)}$ will involve the term $\partial_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)}$ and also a term corresponding to every separate of

$$(10^{\sigma_{10}-\pi_{10}} 01^{\sigma_{01}-\pi_{01}} \dots rs^{\sigma_{rs}-\pi_{rs}} \dots),$$

and thus altogether it will involve

$$(\sigma_{10}-\pi_{10}+1)(\sigma_{01}-\pi_{01}+1)\dots(\sigma_{rs}-\pi_{rs}+1)\dots \text{ terms.}$$

557. The operator $g_{(pq)}$ is a linear function of the operators

$$g_{(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots)},$$

and in the general case this linear function may be regarded as involving as many partition operators as the number (pq) possesses partitions

$$(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots);$$

but for *particular* operands some of these partition operators will be necessarily absent, for a given separable partition may not contain every partition of (pq).

In particular we have in the simplest cases

$$\begin{aligned}
g_{10} &= g_{(10)},\\
g_{01} &= g_{(01)},\\
g_{20} &= g_{(10^2)} - 2g_{(20)},\\
g_{11} &= g_{(10\,01)} - g_{(11)},\\
g_{02} &= g_{(01^2)} - 2g_{(02)},\\
g_{30} &= g_{(10^3)} - 3g_{(20\,10)} + 3g_{(30)},\\
g_{21} &= g_{(10^2 01)} - g_{(20\,01)} - g_{(11\,10)} + g_{(21)},\\
&\text{etc.}
\end{aligned}$$

Some developed expressions of partition operators are

$$\begin{aligned}
g_{(10)} = {} & \partial_{(10)} + (10)\,\partial_{(10^2)} + (01)\,\partial_{(10\,01)} + (20)\,\partial_{(20\,10)} + (10^2)\,\partial_{(10^3)}\\
& + (11)\,\partial_{(11\,10)} + (10\,01)\,\partial_{(10^2 01)} + (02)\,\partial_{(10\,02)} + (01^2)\,\partial_{(10\,01^2)} + \dots;\\
g_{(20\,01)} = {} & \partial_{(20\,01)} + (10)\,\partial_{(20\,10\,01)} + (01)\,\partial_{(20\,01^2)} + (20)\,\partial_{(20^2\,01)} + (10^2)\,\partial_{(20\,10^2 01)}\\
& + (11)\,\partial_{(20\,11\,01)} + (10\,01)\,\partial_{(20\,10\,01^2)} + (02)\,\partial_{(20\,02\,01)} + (01^2)\,\partial_{(20\,01^3)} + \dots.
\end{aligned}$$

558. It will be seen that the operators $g_{(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots)}$ are readily performed upon a separation but their importance lies mainly in important properties which they possess, to which we now proceed. Denote any two operators of the same or different weights

$$g_{(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots)},\quad g_{(10^{\rho_{10}}01^{\rho_{01}}\dots rs^{\rho_{rs}}\dots)},$$

by $g_{(\pi)}$, $g_{(\rho)}$ respectively and $g_{(10^{\pi_{10}+\rho_{10}}01^{\pi_{01}+\rho_{01}}\dots rs^{\pi_{rs}+\rho_{rs}}\dots)}$ by $g_{(\pi+\rho)}$, we have

$$g_{(\pi)}\,g_{(\rho)} = (g_{(\pi)}\,g_{(\rho)}) + g_{(\pi+\rho)};$$

because $g_{(\pi+\rho)}$ is the result of operating with $g_{(\pi)}$ upon the algebraic portion of $g_{(\rho)}$, the differential inverses being regarded as coefficients.

Thence we find

$$g_{(\pi)}\,g_{(\rho)} - g_{(\rho)}\,g_{(\pi)} = 0;$$

establishing that any two partition operators are commutative.

The same fact is expressed by saying that the *alternant* of any two partition operators vanishes.

We learn therefrom that if ϕ be an operand such that

$$g_{(\pi)}\,\phi = 0,$$

then also $g_{(\pi)}\{g_{(\rho)}\,\phi\} = 0$; in other words if ϕ be a solution of the equation $g_{(\pi)} = 0$, then also is $g_{(\rho)}\,\phi$ a solution.

If the right-hand side of the relation

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq} = \Sigma\,\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{rs}!\ldots}\,g_{(10^{\pi_{10}}01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)}$$

be performed upon a linear function of separations of the function

$$(10^{\sigma_{10}}01^{\sigma_{01}}\ldots rs^{\sigma_{rs}}\ldots),$$

which satisfies the equation $g_{pq} = 0$, the effect of the particular partition operator

$$g_{(10^{\pi_{10}}01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)}$$

is the production of terms each of which is a separation of the function

$$(10^{\sigma_{10}-\pi_{10}}01^{\sigma_{01}-\pi_{01}}\ldots rs^{\sigma_{rs}-\pi_{rs}}\ldots);$$

and separations of this partition cannot be produced by any other partition operation included under the sign of summation. In consequence the terms in question must vanish identically and the function which satisfies the equation $g_{pq} = 0$ must also satisfy the equation

$$g_{(10^{\pi_{10}}01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)} = 0.$$

559. We thus obtain the theorem:

"If a function, expressed in terms of separations of a given monomial symmetric function, be annihilated by a certain biweight linear operator, it must also be annihilated by every linear partition operator of the same biweight."

This is the cardinal theorem for use in the calculation of tables of separations of symmetric functions.

Ex. gr. We will calculate the function (31 01) as a linear function of separations of the partition (21 10 01). The law of expressibility shews that this is possible because (21 10) is a partition of (31). Observing that the separation (21) (10) (01) cannot present itself because it is the only separation which on multiplication produces the function (32) we put

$$(31\,01) = A\,(21\,10)(01) + B\,(21\,01)(10) + C\,(10\,01)(21) + D\,(21\,10\,01).$$

Now a monomial symmetric function vanishes by the operation of $g_{(pq)}$ if no partition of the weight pq is comprised amongst its parts. Therefore the only operators which do not annihilate (31 01) are g_{01}, g_{31} and g_{32} and the annihilating operators are g_{10}, g_{21}, ... Hence the partition operators

$$g_{(10)},\ g_{(21)},$$

cause the function to vanish and are sufficient for the purpose.

Retaining only significant terms

$$\{\partial_{(10)} + (01)\,\partial_{(10\,01)} + (21)\,\partial_{(21\,10)} + (21\,01)\,\partial_{(21\,10\,01)}\}\,(31\,01) = 0,$$
$$\{\partial_{(21)} + (10)\,\partial_{(21\,10)} + (01)\,\partial_{(21\,01)} + (10\,01)\,\partial_{(21\,10\,01)}\}\,(31\,01) = 0,$$

leading to $\quad A + C = 0, \quad B + D = 0, \quad C + D = 0, \quad A + B = 0,$

or
$$D = -C = -B = A.$$

Hence

$$(31\,01) = A\,\{(21\,10)(01) - (21\,01)(10) - (10\,01)(21) + (21\,10\,01)\},$$

and now it is easy to see that $A = -\frac{1}{2}$.

The Partition Obliterating Operators.

560. In the foregoing a generalization has been made, in the case of the linear operators $g_{10}, g_{01}, \ldots g_{pq}, \ldots$ from a number to the partition of a number. We proceed *pari passu* with the theory regarding a single system of quantities and now seek the like generalization in regard to the obliterating operators $G_{10}, G_{01}, \ldots G_{pq}, \ldots$. Consider a symmetric function

$$f(a_{10},\ a_{01},\ \ldots a_{pq},\ \ldots) = f$$

to be the product of m functions and write

$$f = f_1 f_2 \ldots f_m.$$

Supposing a_{pq} to be changed into $a_{pq} + \mu a_{p-1,q} + \nu a_{p,q-1}$ we have from previous work

$$\begin{aligned}
&(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f \\
= \;&(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_1 \\
\times\;&(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_2 \\
\times\;&(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_3 \\
\times\;&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\times\;&(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_m.
\end{aligned}$$

Expanding the right-hand side and then equating coefficients of like products of powers of μ and ν we obtain the relations

$$\begin{aligned}
G_{10} f &= \Sigma\,(G_{10} f_1) f_2 f_3 \ldots f_m, \\
G_{20} f &= \Sigma\,(G_{10} f_1)(G_{10} f_2) f_3 \ldots f_m + \Sigma\,(G_{20} f_1) f_2 \ldots f_m, \\
G_{11} f &= \Sigma\,(G_{10} f_1)(G_{01} f_2) f_3 \ldots f_m + \Sigma\,(G_{11} f_1) f_2 \ldots f_m, \\
G_{30} f &= \Sigma\,(G_{10} f_1)(G_{10} f_2)(G_{10} f_3) f_4 \ldots f_m + \Sigma\,(G_{20} f_1)(G_{10} f_2) f_3 \ldots f_m \\
&\quad + \Sigma\,(G_{30} f_1) f_2 f_3 \ldots f_m, \\
G_{21} f &= \Sigma\,(G_{10} f_1)(G_{10} f_2)(G_{01} f_3) f_4 \ldots f_m + \Sigma\,(G_{11} f_1)(G_{10} f_2) f_3 \ldots f_m \\
&\quad + \Sigma\,(G_{20} f_1)(G_{01} f_2) f_3 \ldots f_m + \Sigma\,(G_{21} f_1) f_2 \ldots f_m,
\end{aligned}$$

etc.

Herein the summations are in regard to the different terms that may be obtained by permutation of the m functions $f_1 f_2 \dots f_m$ and there is no diminution in the number of terms arising from any equalities between the m functions. Regarded in another manner we may consider the m functions to be fixed in order and the sum to arise by distributing the operators in all possible ways.

Generally in the expression for $G_{pq} f$ there will occur a summation corresponding to every partition of pq. If such a partition be $(p_1 q_1, p_2 q_2, \dots p_s q_s)$ the summation is

$$\Sigma (G_{p_1 q_1} f_1)(G_{p_2 q_2} f_2) \dots (G_{p_s q_s} f_s) \cdot f_{s+1} \dots f_m.$$

561. Thus when performed upon a product of functions the operation G_{pq} breaks up into as many distinct operations as the weight pq possesses partitions. We write therefore

$$\Sigma (G_{p_1 q_1} f_1)(G_{p_2 q_2} f_2) \dots (G_{p_s q_s} f_s) f_{s+1} \dots f_m = G_{(p_1 q_1\, p_2 q_2 \dots p_s q_s)},$$

and call $G_{(p_1 q_1\, p_2 q_2 \dots p_s q_s)}$ a partition obliterating operator.

In this notation we have

$$G_{10} = G_{(10)},$$
$$G_{20} = G_{(20)} + G_{(10^2)},$$
$$G_{11} = G_{(11)} + G_{(10\,01)},$$
$$G_{30} = G_{(30)} + G_{(20\,10)} + G_{(10^3)},$$
$$G_{21} = G_{(21)} + G_{(11\,10)} + G_{(20\,01)} + G_{(10^2\,01)},$$
etc.

and generally

$$G_{pq} = \Sigma G_{(p_1 q_1 p_2 q_2 \dots p_s q_s)},$$

where the summation is in regard to every partition of pq.

This is a theorem for operating with G_{pq} upon a product of functions and is strictly analogous to that obtained for a single system of quantities. The reader will have no difficulty in seeing that when there are more than two systems of quantities

$$G_{pqr\dots} = \Sigma G_{(p_1 q_1 r_1 \dots,\, p_2 q_2 r_2 \dots,\, \dots\, p_s q_s r_s \dots)}.$$

562. The next step is to connect the linear partition operators $g_{(p_1 q_1 p_2 q_2 \dots p_s q_s)}$ with the partition obliterating operators $G_{(p_1 q_1\, p_2 q_2 \dots p_s q_s)}$.

We have the three relations

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} g_{pq} = \sum_\pi \frac{(-)^{\Sigma\pi - 1} (\Sigma\pi - 1)!}{\pi_{10}!\, \pi_{01}! \dots \pi_{rs}! \dots} g_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)},$$

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\, q!} g_{pq} = \sum_\pi \frac{(-)^{\Sigma\pi - 1} (\Sigma\pi - 1)!}{\pi_{10}!\, \pi_{01}! \dots \pi_{rs}! \dots} G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \dots G_{rs}^{\pi_{rs}} \dots,$$

$$G_{pq} = \sum_\pi G_{(10^{\pi_{10}} 01^{\pi_{01}} \dots rs^{\pi_{rs}} \dots)},$$

the summation in each case being for every partition $(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots)$ of pq. From these we can eliminate the two operators g_{pq}, G_{pq}. Thus the first two relations yield

$$\sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} g_{(10^{\pi_{10}}01^{\pi_{01}}\dots rs^{\pi_{rs}}\dots)} = \sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\dots\pi_{rs}!\dots} G_{10}^{\pi_{10}} G_{01}^{\pi_{01}}\dots G_{rs}^{\pi_{rs}}\dots,$$

and with this we have now merely to combine the third relation. The simplest particular cases are

$$\begin{cases} g_{(10)} = G_{10} = G_{(10)}, \\ g_{(01)} = G_{01} = G_{(01)}, \end{cases}$$

$$\begin{cases} g_{(10^2)} - 2g_{(20)} = G_{10}^2 - 2G_{20} = \{G_{(10)}^2 - 2G_{(10^2)}\} - 2G_{(20)}, \\ g_{(10\,01)} - g_{(11)} = G_{10}G_{01} - G_{11} = \{G_{(10)}G_{(01)} - G_{(10\,01)}\} - G_{(11)}, \\ g_{(01^2)} - 2g_{(02)} = G_{01}^2 - 2G_{02} = \{G_{(01)}^2 - 2G_{(01^2)}\} - 2G_{(02)}, \end{cases}$$

$$\begin{cases} g_{(10^3)} - 3g_{(20\,10)} + 3g_{(30)} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30} \\ \qquad = \{G_{(10)}^3 - 3G_{(10^2)}G_{(10)} + 3G_{(10^3)}\} - 3\{G_{(20)}G_{(10)} - G_{(20\,10)}\} + 3G_{(30)}, \\ g_{(10^2\,01)} - g_{(20\,01)} - g_{(11\,10)} + g_{(21)} = G_{10}^2 G_{01} - G_{20}G_{01} - G_{11}G_{10} + G_{21} \\ \qquad = \{G_{(10)}^2 G_{(01)} - G_{(10^2)}G_{(01)} - G_{(10\,01)}G_{(10)} + G_{(10^2\,01)}\} - \{G_{(20)}G_{(01)} - G_{(20\,01)}\} \\ \qquad - \{G_{(11)}G_{(10)} - G_{(11\,10)}\} + G_{(21)}, \end{cases}$$

and we learn much from their examination.

563. Take the result

$$g_{(10^2)} - 2g_{(20)} = \{G_{(10)}^2 - 2G_{(10^2)}\} - 2G_{(20)}$$

into consideration.

Suppose that the operand is a linear function of separations of a given partition.

Upon it the two sides of the operator relation must produce identically the same function; but the terms produced by $g_{(10^2)}$, $G_{(10)}^2$ and $G_{(10^2)}$ are separations of one partition, and the terms produced by $g_{(20)}$ and $G_{(20)}$ are separations of one partition *different* from the former. Hence for the given operand the relation breaks up into the two relations

$$g_{(10^2)} = G_{(10)}^2 - 2G_{(10^2)},$$
$$g_{(20)} = G_{(20)}.$$

This reasoning is applicable in general and gives us the relations

$$g_{(10)} = G_{(10)},$$

$$\begin{cases} g_{(10^2)} = G_{(10)}^2 - 2G_{(10^2)}, \\ g_{(20)} = G_{(20)}, \end{cases}$$

$$\begin{cases} g_{(10\,01)} = G_{(10)}\,G_{(01)} - G_{(10\,01)}, \\ g_{(11)} \quad = G_{(11)}, \end{cases}$$

$$\begin{cases} g_{(10^3)} \quad = G^3_{(10)} - 3G_{(10^2)}\,G_{(10)} + 3G_{(10^3)}, \\ g_{(20\,10)} = G_{(20)}\,G_{(10)} - G_{(20\,10)}, \\ g_{(30)} \quad = G_{(30)}, \end{cases}$$

$$\begin{cases} g_{(10^2 01)} = G^2_{(10)}\,G_{(01)} - G_{(10^2)}\,G_{(01)} - G_{(10\,01)}\,G_{(10)} + G_{(10^2 01)}, \\ g_{(20\,01)} = G_{(20)}\,G_{(01)} - G_{(20\,01)}, \\ g_{(11\,10)} = G_{(11)}\,G_{(10)} - G_{(11\,10)}, \\ g_{(21)} \quad = G_{(21)}. \end{cases}$$

564. The general formula that will be established is

$$\frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\,\ldots\,\pi_{rs}!\,\ldots}\,g_{(10^{\pi_{10}}\,01^{\pi_{01}}\,\ldots\,rs^{\pi_{rs}}\,\ldots)} = \sum_j \frac{(-)^{\Sigma j-1}\,(\Sigma j-1)!}{j_1!\,j_2!\,\ldots}\,G^{j_1}_{(J_1)}\,G^{j_2}_{(J_2)}\,\ldots,$$

where the summation is for every separation

$$(J_1)^{j_1}\,(J_2)^{j_2}\,\ldots$$

of the partition

$$(10^{\pi_{10}}\,01^{\pi_{01}}\,\ldots\,rs^{\pi_{rs}}\,\ldots).$$

We proceed from the formula

$$\sum_\pi (-)^{\Sigma\pi-1}\,\frac{(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\,\ldots\,\pi_{rs}!\,\ldots}\,g_{(10^{\pi_{10}}\,01^{\pi_{01}}\,\ldots\,rs^{\pi_{rs}})}$$

$$= \sum_\pi (-)^{\Sigma\pi-1}\,\frac{(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\,\ldots\,\pi_{rs}!\,\ldots}\,G^{\pi_{10}}_{10}\,G^{\pi_{01}}_{01}\,\ldots\,G^{\pi_{rs}}_{rs}\,\ldots$$

and we have to substitute for G_{10}, G_{01}, ... G_{rs}, ... the sums of the partition operators of weights 10, 01, ... rs, ... respectively. We have then to collect on the right all the G operator products which are associated with separations of the same partition and to equate them to the corresponding g operators on the left.

Suppose $\qquad (\sigma_1),\ (\sigma_2),\ \ldots$

to be the several partitions of (rs); we find that

$$\frac{1}{\pi_{rs}!}\,G^{\pi_{rs}}_{rs} = \frac{1}{\pi_{rs}!}\,\{G_{(\sigma_1)} + G_{(\sigma_2)} + \ldots\}^{\pi_{rs}} = \Sigma\,\frac{G^{\pi_{\sigma_1}}_{(\sigma_1)}\,G^{\pi_{\sigma_2}}_{(\sigma_2)}\,\ldots}{\pi_{\sigma_1}!\,\pi_{\sigma_2}!\,\ldots},$$

leading to the relation

$$(-)^{\Sigma\pi-1}\,\frac{(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\,\ldots\,\pi_{rs}!\,\ldots}\,g_{(10^{\pi_{10}}\,01^{\pi_{01}}\,\ldots\,rs^{\pi_{rs}}\,\ldots)} = \sum_j (-)^{\Sigma j-1}\,\frac{(\Sigma j-1)!}{j_1!\,j_2!\,\ldots}\,G^{j_1}_{(J_1)}\,G^{j_2}_{(J_2)}\,\ldots$$

where $\qquad (J_1)^{j_1}\,(J_2)^{j_2}\,\ldots$

is a separation of the symmetric function

$$(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)$$

and the summation is in regard to the whole of such separations.

This formula is to be compared with that which expresses s_{pq} in terms of separations of any partition

$$(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)$$

of the bipartite number pq, viz.:

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{rs}!\ldots}s_{(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)}=\sum_j(-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots.$$

565. From the above relations we can express the partition G operators in terms of the partition g operators. Thus

$$G_{(10)}=g_{(10)},$$

$$\left\{\begin{aligned}2!\,G_{(10^2)}&=g_{(10)}^2-g_{(10^2)},\\ G_{(20)}&=g_{(20)},\end{aligned}\right.$$

$$\left\{\begin{aligned}G_{(10\,01)}&=g_{(10)}g_{(01)}-g_{(10\,01)},\\ G_{(11)}&=g_{(11)},\end{aligned}\right.$$

$$\left\{\begin{aligned}3!\,G_{(10^3)}&=g_{(10)}^3-3g_{(10^2)}\,g_{(10)}+2g_{(10^3)},\\ G_{(20\,10)}&=g_{(20)}\,g_{(10)}-g_{(20\,10)},\\ G_{(30)}&=g_{(30)},\end{aligned}\right.$$

$$\left\{\begin{aligned}2G_{(10^2\,01)}&=g_{(10)}^2\,g_{(01)}-g_{(10^2)}\,g_{(01)}-2g_{(10\,01)}\,g_{(10)}+2g_{(10^2\,01)},\\ G_{(20\,01)}&=g_{(20)}\,g_{(01)}-g_{(20\,01)},\\ G_{(11\,10)}&=g_{(11)}\,g_{(10)}-g_{(11\,10)},\\ G_{(21)}&=g_{(21)}.\end{aligned}\right.$$

The general formula that is established follows the law by which any symmetric function is expressed in terms of the one-part symmetric functions. Thus the first of the last batch of formulæ is in correspondence with the formula

$$2\,(10^2\,01)=s_{10}^2s_{01}-s_{20}s_{01}-2s_{11}s_{10}+2s_{21},$$

and it will be noted that

$$s_{\Sigma p,\,\Sigma q}$$

takes the place of $g_{(p_1q_1\,p_2q_2\,p_3q_3\ldots)}$ on the right-hand side.

It will be noted that the partitions associated with the right-hand side of the algebraic formula, viz.:

$$(10^2\,01),\quad(20\,01),\quad(11\,10),\quad(21),$$

are the specifications of the separations

$$(10)^2(01),\quad (10^2)(01),\quad (10\,01)(10),\quad (10^2\,01),$$

which are associated with the right-hand side of the operator formula.

The formula which expresses any monomial symmetric function in terms of functions with a single part in their partitions is

$$(-)^{\Sigma\pi-1}(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)$$
$$=\Sigma\,(-)^{\Sigma j-1}\frac{(\Sigma\rho_1-1)!\,(\Sigma\rho_2-1)!\ldots}{j_1!\,j_2!\ldots\rho_{11}!\,\rho_{12}!\ldots\rho_{21}!\,\rho_{22}!\ldots\;\ldots\ldots}\,s_{(J_1)}^{j_1}\,s_{(J_2)}^{j_2}\ldots,$$

where $\quad (J_1)=(p_{11}q_{11}^{\rho_{11}}\,p_{12}q_{12}^{\rho_{12}}\ldots),\quad (J_2)=(p_{21}q_{21}^{\rho_{21}}\,p_{22}q_{22}^{\rho_{22}}\ldots),\;\ldots\ldots$

and $(J_1)^{j_1}(J_2)^{j_2}\ldots$ is a separation of

$$(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots).$$

If $(J_1), (J_2), \ldots$ are of weights $p_1q_1, p_2q_2, \ldots$ respectively, $s_{(J_1)}, s_{(J_2)}, \ldots$ are equal to $s_{p_1q_1}, s_{p_2q_2}, \ldots$ respectively.

The corresponding operator formula will be found to be

$$(-)^{\Sigma\pi-1}\,G_{(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)}$$
$$=\Sigma\,(-)^{\Sigma j-1}\frac{(\Sigma\rho_1-1)!\,(\Sigma\rho_2-1)!\ldots}{j_1!\,j_2!\ldots\rho_{11}!\,\rho_{12}!\ldots\rho_{21}!\,\rho_{22}!\ldots\;\ldots\ldots}\,g_{(J_1)}^{j_1}\,g_{(J_2)}^{j_2}\ldots,$$

but here of course we are not at liberty to substitute $p_1q_1, p_2q_2, \ldots$ for $(J_1), (J_2), \ldots$ respectively in the formula.

566. In these formulæ the multiplications of operations denote successive operations. The products of g partition operators are readily transformed, so that the multiplications may be symbolic; for, denoting symbolic multiplication by external brackets as usual, we find

$$2!\,G_{(10^2)}=(g_{(10)}^2),$$
$$G_{(10\,01)}=(g_{(10)}\,g_{(01)}),$$
$$3!\,G_{(10^3)}=(g_{(10)}^3),$$

and in general

$$\pi_{10}!\,\pi_{01}!\ldots\pi_{rs}!\ldots G_{(10^{\pi_{10}}\,01^{\pi_{01}}\ldots rs^{\pi_{rs}}\ldots)}=(g_{(10)}^{\pi_{10}}\,g_{(01)}^{\pi_{01}}\ldots g_{(rs)}^{\pi_{rs}}\ldots);$$

and this is to be compared with a previous result

$$p!\,q!\,G_{pq}=(g_{10}^p\,g_{01}^q).$$

These results are readily generalizable for multipartite symmetric functions in general. Thus

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{100\ldots}!\,\pi_{010\ldots}!\ldots\pi_{rst\ldots}!\ldots}\,g_{(100\ldots^{\pi_{100\ldots}}\,010\ldots^{\pi_{010\ldots}}\ldots\ldots\,rst\ldots^{\pi_{rst\ldots}})}$$
$$=\sum_j\frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\ldots}\,G_{(J_1)}^{j_1}\,G_{(J_2)}^{j_2}\ldots,$$

where $(J_1)^{j_1}(J_2)^{j_2}\ldots$ is a separation of $(100\ldots^{\pi_{100\ldots}}\,010\ldots^{\pi_{010\ldots}}\ldots\ldots rst\ldots^{\pi_{rst\ldots}}\ldots)$, and the summation is in regard to the whole of the separations.

The reader will have no difficulty in reversing this formula.

Also

$$\pi_{100\ldots}!\,\pi_{010\ldots}!\ldots\pi_{rst\ldots}!\ldots G_{(100\ldots^{\pi_{100\ldots}}\,010\ldots^{\pi_{010\ldots}}\ldots\ldots rst\ldots^{\pi_{rst\ldots}}\ldots)}$$

$$=(g^{\pi_{100\ldots}}_{(100\ldots)}\,g^{\pi_{010\ldots}}_{(010\ldots)}\ldots g^{\pi_{rst\ldots}}_{(rst\ldots)}\ldots).$$

These relations are of importance in the studies of combinations on a chess board or lattice, which will be taken up in the next chapter. In that theory it will be found that certain enumerations are determined in the forms

$$G_{11}^2\,a_{11}^2,\quad G_{111}^3\,a_{111}^3,\quad G_{1111}^4\,a_{1111}^4,\ldots\ldots$$

and generally in the form $G_{pqr\ldots}^{p+q+r+\ldots}\,a_{pqr\ldots}^{p+q+r+\ldots}$.

In fact these expressions, on evaluation, are integers, and severally enumerate interesting combinations on a chess board.

567. We have above the results that have been, or are likely to be, of the greatest service in Combinatory Analysis. An example may be given of the use of the operators in the multiplication of symmetric functions.

Suppose that we require the coefficient of (11^2) in the product

$$(10^2)(01)^2.$$

Put $$(10^2)(01)^2=\ldots+A(11^2)+\ldots.$$

When we operate with G_{11}^2 on the right-hand side the result is A, since every other term is caused to vanish.

Hence $$A=G_{11}^2\,(10^2)(01)^2.$$

Now $$G_{11}=G_{(11)}+G_{(10\,01)},$$

and since the operand involves no part (11) we find

$$A=G_{(10\,01)}^2\,(10^2)(01)^2=2G_{(10\,01)}\,(10)(01)=2.$$

Again, to find the coefficient of (11 10 01) in the product

$$(10\,01)(10)(01),$$

we have to evaluate

$$\begin{aligned}&G_{11}\,G_{10}\,G_{01}\,(10\,01)(10)(01)\\ &\equiv G_{(10\,01)}\,G_{(10)}\,G_{(01)}\,(10\,01)(10)(01)\\ &=G_{(10\,01)}\,G_{(10)}\,\{(10\,01)(10)+(10)^2(01)\}\\ &=G_{(10\,01)}\,\{(10\,01)+(10)(01)+2(10)(01)\},\end{aligned}$$

and remembering that $G_{(10\,01)}(10\,01)=0$, since in operating with $G_{(10\,01)}$ the parts (10), (01) must be taken away from separate factors of a product, we find the result

$$G_{(10\,01)}\,.\,3(10)(01)=3.$$

The Construction of Symmetrical Tables.

n established in Art. 542 that it is possible to form two n connexion with every partition of every biweight. tables are given as far as weight 4 inclusive. Of that biweights 40, 04 as they are practically the same as system of quantities. Of the remaining biweights essary to give the two former, because the biweights he same.

artitions of biweight 31, viz.:

$(10^3\,01)$, $(11\,10^2)$, $(20\,10\,01)$, $(20\,11)$, $(21\,10)$, $(30\,01)$, (31),

and nine of biweight 22, viz.:

$(10^2\,01^2)$, $(11\,10\,01)$, (11^2), $(20\,01^2)$, $(20\,02)$, $(21\,01)$, (22).
$(02\,10^2)$, $(12\,10)$,

Of these the tables for $(20\,01^2)$, $(21\,01)$ give those for $(02\,10^2)$, $(12\,10)$ respectively by transposing the elements of the biparts, and the four corresponding to (31) and (22) are mere identities, so that the number is reduced from 32 to 24.

The earlier tables which are necessary are those of the partitions $(10\,01)$, $(20\,01)$, $(11\,10)$, $(10^2\,01)$.

Each table is read horizontally, or as may be said, by rows and not by columns.

CHAPTER V

FURTHER THEORY OF THE LATIN SQUARE

569. The question of the Latin Square can be regarded from another point of view. Let us consider the symmetric functions of many systems of quantities and the associated differential operators. Three systems of quantities may be taken as typical of the general case. As operand we may have the product

$$a_{\lambda_1\mu_1\nu_1}a_{\lambda_2\mu_2\nu_2}\dots a_{\lambda_s\mu_s\nu_s}\equiv(100^{\lambda_1}010^{\mu_1}001^{\nu_1})(100^{\lambda_2}010^{\mu_2}001^{\nu_2})\dots(100^{\lambda_s}010^{\mu_s}001^{\nu_s})$$

and as operator

$$G_{p_1q_1r_1}G_{p_2q_2r_2}\dots G_{p_tq_tr_t},$$

$(\lambda_1\mu_1\nu_1\lambda_2\mu_2\nu_2\dots\lambda_s\mu_s\nu_s)$ and $(p_1q_1r_1p_2q_2r_2\dots p_tq_tr_t)$ being partitions of the same tripartite number.

It will be remembered from Chapter III that

$$G_{pqr}=\frac{1}{p!}\frac{1}{q!}\frac{1}{r!}(g_{100}^p g_{010}^q g_{001}^r),$$

where

$$g_{100}=\Sigma a_{p-1\,q\,r}\,\partial_{a_{pqr}},$$

$$g_{010}=\Sigma a_{p\,q-1\,r}\,\partial_{a_{pqr}},$$

$$g_{001}=\Sigma a_{p\,q\,r-1}\,\partial_{a_{pqr}}.$$

Moreover it was proved that if on multiplication

$$a_{\lambda_1\mu_1\nu_1}a_{\lambda_2\mu_2\nu_2}\dots a_{\lambda_s\mu_s\nu_s}=\dots+A\,(p_1q_1r_1p_2q_2r_2\dots p_tq_tr_t)+\dots,$$

then

$$G_{p_1q_1r_1}G_{p_2q_2r_2}\dots G_{p_tq_tr_t}a_{\lambda_1\mu_1\nu_1}a_{\lambda_2\mu_2\nu_2}\dots a_{\lambda_s\mu_s\nu_s}=A.$$

We must carry out the operation upon the function

$$(100^{\lambda_1}010^{\mu_1}001^{\nu_1})(100^{\lambda_2}010^{\mu_2}001^{\nu_2})\dots(100^{\lambda_s}010^{\mu_s}001^{\nu_s})$$

and then determine the nature of the diagrams enumerated by the number A. Consider first the operation of $G_{p_1q_1r_1}$. It operates through the partitions of $(p_1q_1r_1)$ upon the operand. The only effective partition is

$$(100^{p_1}010^{q_1}001^{r_1}),$$

and its parts must be picked out from the factors of the operand in all possible ways subject to the condition that only one part must be taken from any one factor. The operation breaks up as usual into a number of minor operations and each of these yields a first row of a lattice of s columns,

s being the number of factors of the operand, and of t rows, t being the number of factors composing the operator. The first row comprises s compartments and in p_1, q_1, r_1 of these will occur the tripartite numbers 100, 010, 001 respectively, $p_1+q_1+r_1$ compartments being occupied and $s-p_1-q_1-r_1$ unoccupied.

Read from left to right these tripartite numbers constitute a composition into parts 100, 010, 001 of the tripartite number $(p_1q_1r_1)$. Similarly a minor operation of $G_{p_2q_2r_2}$ gives in each case a second row involving the numbers 100, 010, 001, p_2, q_2, r_2 times respectively, and these numbers read from left to right constitute a composition into parts 100, 010, 001 of the tripartite number $(p_2q_2r_2)$.

We finally arrive at a diagram such that the tripartite numbers in the successive rows read from left to right constitute compositions into parts 100, 010, 001 of the tripartite numbers $(p_1q_1r_1)$, $(p_2q_2r_2)$, ... $(p_tq_tr_t)$, respectively; and in the successive columns if the reading be from top to bottom, we have compositions into parts 100, 010, 001, of the tripartite numbers $(\lambda_1\mu_1\nu_1)$, $(\lambda_2\mu_2\nu_2)$, ... $(\lambda_s\mu_s\nu_s)$ respectively. To give the proposition a purely literal form we write 100, 010, 001 equal to a, b, c respectively, and the diagrams are then such that there are s columns and t rows, the products of letters in the successive rows being

$$a^{p_1}b^{q_1}c^{r_1},\quad a^{p_2}b^{q_2}c^{r_2}, \dots\ a^{p_t}b^{q_t}c^{r_t},$$

and in the successive columns

$$a^{\lambda_1}b^{\mu_1}c^{\nu_1},\quad a^{\lambda_2}b^{\mu_2}c^{\nu_2}, \dots\ a^{\lambda_s}b^{\mu_s}c^{\nu_s}.$$

The number of these diagrams is therefore

$$G_{p_1q_1r_1}\,G_{p_2q_2r_2}\dots G_{p_tq_tr_t}\,a_{\lambda_1\mu_1\nu_1}\,a_{\lambda_2\mu_2\nu_2}\dots a_{\lambda_s\mu_s\nu_s}.$$

In this distribution not more than one letter occurs in each compartment and there may be empty compartments.

Ex. gr. With the operator $G_{111}\,G_{101}\,G_{111}$ and the operand $a_{110}\,a_{101}\,a_{010}\,a_{102}$ we obtain the integer 7 which enumerates diagrams of four columns and three rows such that the products of letters in the successive rows are abc, ac, abc and in the successive columns ab, ac, b, ac^2. The seven diagrams are

a	0	b	c
0	a	0	c
b	c	0	a

a	0	b	c
0	c	0	a
b	a	0	c

0	a	b	c
a	0	0	c
b	c	0	a

b	a	0	c
0	c	0	a
a	0	b	c

0	c	b	a
a	0	0	c
b	a	0	c

b	c	0	a
a	0	0	c
0	a	b	c

b	c	0	a
0	a	0	c
a	0	b	c

and they are the only ones satisfying the conditions.

If we rotate these diagrams we obtain those derivable from

$$G_{110}\,G_{101}\,G_{010}\,G_{102}\,a_{111}\,a_{101}\,a_{111},$$

which therefore necessarily has the value 7. In general we thus prove that

$$G_{p_1q_1r_1}\,G_{p_2q_2r_2}\ldots G_{p_tq_tr_t}\,a_{\lambda_1\mu_1\nu_1}\,a_{\lambda_2\mu_2\nu_2}\ldots a_{\lambda_s\mu_s\nu_s}$$
$$= G_{\lambda_1\mu_1\nu_1}\,G_{\lambda_2\mu_2\nu_2}\ldots G_{\lambda_s\mu_s\nu_s}\,a_{p_1q_1r_1}\,a_{p_2q_2r_2}\ldots a_{p_tq_tr_t},$$

involving a law of symmetry already established in this Section, Art. 542.

570. If we require diagrams with no empty compartments we must have

$$p_1+q_1+r_1=p_2+q_2+r_2=\ldots=p_t+q_t+r_t=s,$$

and
$$\lambda_1+\mu_1+\nu_1=\lambda_2+\mu_2+\nu_2=\ldots=\lambda_s+\mu_s+\nu_s=t.$$

Thus
$$G_{211}\,G_{121}\,a_{110}\,a_{101}\,a_{011}\,a_{110}$$

leads to the three diagrams

a	a	c	b
b	c	b	a

a	c	b	a
b	a	c	b

b	a	c	a
a	c	b	b

possessing the property that the letters in the successive rows make the products a^2bc, ab^2c and in the successive columns the products ab, ac, bc, ab.

These diagrams when rotated correspond to the same integer 3, viz.:

$$G_{110}\,G_{101}\,G_{011}\,G_{110}\,a_{211}\,a_{121}.$$

Further if the diagrams are to be squares we must have $s=t$. Moreover if the diagrams are to be Latin Squares the product connected with each row and each column must be abc and for the third order we find that the Latin Squares are enumerated by

$$G_{111}^3\,a_{111}^3,$$

and for the order n by

$$G_{111\ldots1}^n\,a_{111\ldots1}^n,$$

a very simple expression for the numbers.

571. Again corresponding to

$$G_{pqr}^{\lambda+\mu+\nu}\,a_{\lambda\mu\nu}^{p+q+r}$$

we have diagrams such that there are $\lambda+\mu+\nu$ rows and $p+q+r$ columns with a product $a^pb^qc^r$ corresponding to each row and $a^\lambda b^\mu c^\nu$ to each column.

Ex. gr. The expression $G_{222}^3\,a_{111}^6$ gives diagrams such as

a	b	a	c	c	b
c	a	c	b	b	a
b	c	b	a	a	c

which have the property that each row gives the product $a^2b^2c^2$ and each column the product abc.

In particular

$$G_{\lambda\mu\nu}^{\lambda+\mu+\nu}\, a_{\lambda\mu\nu}^{\lambda+\mu+\nu}$$

gives square diagrams of order $\lambda+\mu+\nu$ such that the product $a^\lambda b^\mu c^\nu$ is given by each row and each column.

In general

$$G_{\lambda\mu\nu\ldots}^{\lambda+\mu+\nu+\ldots}\, a_{\lambda\mu\nu\ldots}^{\lambda+\mu+\nu+\ldots}$$

gives square diagrams of order $\lambda+\mu+\nu+\ldots$ such that the product corresponding to each row and each column is $a^\lambda b^\mu c^\nu\ldots$.

This is the generalization of the Latin Square previously met with in Section V, Art. 219.

TABLES OF SYMMETRIC FUNCTIONS OF TWO SYSTEMS OF QUANTITIES

Biweight 11. Partition (10 01).

	(10 01)	(10) (01)
(11)	−1	1
(10 01)	1	

	(11)	(10 01)
(10 01)		1
(10) (01)	1	1

Biweight 21. Partition (20 01).

	(20 01)	(20) (01)
(21)	−1	1
(20 01)	1	

	(21)	(20 01)
(20 01)		1
(20) (01)	1	1

Partition (11 10).

	(11 10)	(11) (10)
(21)	−1	1
(11 10)	1	

	(21)	(11 10)
(11 10)		1
(11) (10)	1	1

Partition (10^2 01).

	a_{21}	$a_{20}\,a_{01}$	$a_{11}\,a_{10}$	$a_{10}^2\,a_{01}$
(21)	1	−1	−1	1
(20 01)	−1	−1	1	
(11 10)	−1	1		
(10^2 01)	1			

	(21)	(20 01)	(11 10)	(10^2 01)
a_{21}				1
$a_{20}\,a_{01}$			1	1
$a_{11}\,a_{10}$		1	1	2
$a_{10}^2\,a_{01}$	1	1	2	2

Biweight 31. Partition (30 01).

	(30 01)	(30) (01)
(31)	−1	1
(30 01)	1	

	(31)	(30 01)
(30 01)		1
(30) (01)	1	1

Partition (21 10).

	(21 10)	(21) (10)
(31)	−1	1
(21 10)	1	

	(31)	(21 10)
(21 10)		1
(21) (10)	1	1

Partition (20 11).

	(20 11)	(20) (11)
(31)	−1	1
(20 11)	1	

	(31)	(20 11)
(20 11)		1
(20) (11)	1	1

Partition (20 10 01).

	(20 10 01)	(20 10) (01)	(20 01) (10)	(10 01) (20)	(20) (10) (01)
(31)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
(30 01)	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
(21 10)	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
(20 11)	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	
(20 10 01)	1				

	(31)	(30 01)	(21 10)	(20 11)	(20 10 01)
(20 10 01)					1
(20 10) (01)			1	1	1
(20 01) (10)		1		1	1
(10 01) (20)		1	1		1
(20) (10) (01)	1	1	1	1	1

Partition (11 10²).

	(11 10²)	(10²) (11)	(11 10) (10)	(11) (10)²
(31)	1	−1	−1	1
(20 11)	−1	−1	1	
(21 10)	−1	1		
(11 10²)	1			

	(31)	(20 11)	(21 10)	(11 10²)
(11 10²)				1
(10²) (11)			1	1
(11 10) (10)		1	1	2
(11) (10)²	1	1	2	2

Partition (10³ 01).

	a_{31}	$a_{30}a_{01}$	$a_{21}a_{10}$	$a_{20}a_{11}$	$a_{20}a_{10}a_{01}$	$a_{11}a_{10}^2$	$a_{10}^3a_{01}$
(31)	−1	1	1	1	−2	−1	1
(30 01)	1	2	−1	−1	−1	1	
(21 10)	1	−1	0	−1	1		
(20 11)	1	−1	−1	1			
(20 10 01)	−2	−1	1				
(11 10²)	−1	1					
(10³ 01)	1						

	(31)	(30 01)	(21 10)	(20 11)	(20 10 01)	(11 10²)	(10³ 01)
a_{31}							1
$a_{30}a_{01}$						1	1
$a_{21}a_{10}$					1	1	3
$a_{20}a_{11}$				1	1	2	3
$a_{20}a_{10}a_{01}$			1	1	1	3	3
$a_{11}a_{10}^2$		1	1	2	3	4	6
$a_{10}^3a_{01}$	1	1	3	3	3	6	6

BIWEIGHT 22. Partition (21 01).

	(21 01)	(21) (01)
(22)	−1	1
(21 01)	1	

	(22)	(21 01)
(21 01)		1
(21) (01)	1	1

Partition (20 02).

	(20 02)	(20) (02)
(22)	−1	1
(20 02)	1	

	(22)	(20 02)
(20 02)		1
(20) (02)	1	1

Partition (11^2).

	(11^2)	$(11)^2$
(22)	−2	1
(11^2)	1	

	(22)	(11^2)
(11^2)		1
$(11)^2$	1	2

Partition ($20\ 01^2$).

	($20\ 01^2$)	(20) (01^2)	(20 01) (01)	(20) $(01)^2$
(22)	1	−1	−1	1
(20 02)	−1	−1	1	
(21 01)	−1	1		
($20\ 01^2$)	1			

	(22)	(20 02)	(21 01)	($20\ 01^2$)
($20\ 01^2$)				1
(20) (01^2)			1	1
(20 01) (01)		1	1	2
(20) $(01)^2$	1	1	2	2

Partition (11 10 01).

	(11 10 01)	(11 10) (01)	(11 01) (10)	2 (10 01) (11)	(11) (10) (01)
(22)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	1
(21 01)	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	
(12 10)	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	
(11^2)	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	
(11 10 01)	1				

	(22)	(21 01)	(12 10)	(11^2)	(11 10 01)
(11 10 01)					1
(11 10) (01)			1	2	1
(11 01) (10)		1	0	2	1
2 (10 01) (11)		2	2	0	2
(11) (10) (01)	1	1	1	2	1

Partition ($10^2\ 01^2$).

	a_{22}	$a_{21}a_{01}$	$a_{12}a_{10}$	$a_{20}a_{02}$	a_{11}^2	$a_{20}a_{01}^2$	$a_{02}a_{10}^2$	$a_{11}a_{10}a_{01}$	$a_{10}^2a_{01}^2$
(22)	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	1
(21 01)	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	
(12 10)	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	
(20 02)	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$	
(11^2)	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	
(20 01^2)	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	
(02 10^2)	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	
(11 10 01)	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	
($10^2\ 01^2$)	1								

	(22)	(21 01)	(12 10)	(20 02)	(11^2)	(20 01^2)	(02 10^2)	(11 10 01)	($10^2\ 01^2$)
a_{22}									1
$a_{21}a_{01}$							1	1	2
$a_{12}a_{10}$						1	0	1	2
$a_{20}a_{02}$					1	0	0	1	1
a_{11}^2				1	2	2	2	2	4
$a_{20}a_{01}^2$			1	0	2	0	1	2	2
$a_{02}a_{10}^2$		1	0	0	2	1	0	2	2
$a_{11}a_{10}a_{01}$		1	1	1	2	2	2	3	4
$a_{10}^2a_{01}^2$	1	2	2	1	4	2	2	4	4

Enumeration of Solid Graphs.

Number of Nodes	Number of Distinct Graphs having one aspect	Number of Distinct Graphs having two aspects	Number of Distinct Graphs having three aspects	Number of Distinct Graphs having six aspects	Total Number of Distinct Graphs	Total Number of Graphs
	1	0	0	0	1	1
	0	0	1	0	1	3
	0	0	2	0	2	6
	1	0	2	1	4	13
	0	0	4	2	6	24
	0	0	6	5	11	48
	2	0	6	11	19	86
	1	0	11	21	33	160
	0	0	16	39	55	282
	2	0	20	73	95	500
11	1	0	28	129	158	859
12	0	0	41	226	267	1479
13	2	1	51	388	442	2485
14	1	1	70	659	731	4167
15	0	0	93	1100	1193	6879
16	3	1	122	1821	1947	11297

If we add together the 2nd, 3rd, 4th and 5th columns we obtain the 6th column.

Also adding the 2nd column, twice the 3rd, three times the 4th, and six times the 5th we obtain the 7th column.

The graphs are unrestricted in regard to the numbers of nodes that may be placed along the three axes.

INDEX TO THE TWO VOLUMES

[*Volumes* are indicated by Roman numerals.
Pages are indicated by Arabic numerals.]

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